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GENERALIZED JACOBIANS OF VECTOR-VALUED CONVEX FUNCTIONS

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Generalized Jacobians of Vector-Valued Convex Functions

Teemu Pennanen Jonathan Eckstein

Abstract. We study differential behavior of vector-valued convex functions in finite-dimensional spaces. Convexity of a vector-valued function is defined as the convexity of its epigraph induced by a convex cone in the range space. The Jacobian of a vector-valued function f at a point x is defined to be the sublinear mapping whose graph is the tangent cone of the epigraph of f at the point $(x, f(x))$. Sublinear mappings may be regarded as set-valued generalizations of linear transformations, and as such, they are natural objects to replace classical Jacobians in the cases where they do not exist as linear transformations. Using the theory of sublinear mappings developed by Rockafellar, we show how this approach leads to a natural analogue of the differential calculus in classical analysis, with calculus rules that are more general than those obtained for the subdifferential in convex analysis. A crucial extension is a chain rule of subdifferentiation for a convex composition of convex functions, generalizing the corresponding rule in classical analysis. This rule provides a convenient way to derive optimality conditions for a wide class of convex optimization problems.

1 Introduction

Let X and U be two finite-dimensional real spaces, and f a function from X to U defined on a subset $\operatorname{dom} f \subset X$ called the *domain* of f. Given a convex cone $K \subset U$, the K*-epigraph* of f is the subset of $X \times U$ defined by

$$\operatorname{epi}_K f = \{(x, u) \mid x \in \operatorname{dom} f,\ f(x) - u \in K\}.$$

We will say that f is convex with respect to K, or simply K*-convex,* if $\operatorname{epi}_K f$ is a convex set in $X \times U$. When $U = \Re$ and K is the cone of nonpositive reals, K-convexity is equivalent to convexity of real-valued functions in the usual sense.

K-convex functions arise naturally in many applications and they have been studied, for example, in [9, 8, 13, 18, 7, 4, 16, 3, 17, 5]. Many of these works [8, 18, 7, 16, 3] use ordered vector spaces as their basic framework, thus avoiding explicit reference to the cone K defining convexity. Treating the functions and cones as pairs, we can simultaneously handle functions which are convex with respect to different cones. In general, we do not place any restrictions on the cones other than that they be convex and include the origin. The purpose of this paper is to develop a differential theory for vector-valued convex functions.

In classical analysis, one defines the Jacobian of a function $f : X \to U$ at a point x to be the linear transformation $\nabla f(x) : X \to U$ satisfying

$$f(x') = f(x) + \nabla f(x)(x' - x) + o(x' - x),$$

where o is such that $o(z)/\|z\| \to 0$ as $z \to 0$. Thus, $f(x) + \nabla f(x)(x' - x)$ is a linear approximation of f near x. If $\nabla f(x)$ exists, one says that f is smooth near x. When $\nabla f(x)$ does not exist, one can define more general objects which still contain information about the differential behaviour of f near the point x.

In the presence of convexity, the differential analysis simplifies considerably. The approach convex analysis has adopted in the case of real-valued functions is centered on the notion of the subdifferential. The differential behavior of a real-valued convex function f near a point x is described by a subset $\partial f(x)$ of the dual space X^* of X:

$$\partial f(x) = \{x^* \in X^* \mid f(x') \geq f(x) + \langle x^*, x' - x\rangle\ \forall x' \in \operatorname{dom} f\}.$$

As long as the functions are real-valued, this approach works well, as has been proved by numerous applications since the introduction of subdifferential in early 1960's. In trying to directly generalize this framework to vector-valued functions, one is tempted to define the subdifferential of a K-convex function $f : X \to U$ at a point x as the set of linear transformations

$$\partial f(x) = \{T \in L(X, U) \mid f(x') - f(x) - T(x' - x) \in K\ \forall x' \in \operatorname{dom} f\}.$$

This approach is the most common; see for example [8, 18, 16, 3]. The sharpest results along these lines are given in [7]. Here, as in the case of real-valued functions, we are faced with a *set* in the operator space — a fundamentally different concept from the classical Jacobian, which is an *operator.* Furthermore, as noted on page 75 of [14], convexity is not a property that behaves naturally under matrix multiplications.

We propose a conceptually different approach to the differential theory of K-convex functions, more parallel to that of classical analysis. The linear transformation $\nabla f(x)$ will be replaced by an analogue of a linear transformation in convex analysis, namely by a *sublinear mapping* [15, Section 8G] (a *convex process* in the terminology of [12, Section 39]). Our Jacobian is just the sublinear mapping from X to U whose graph is the tangent cone to the K-epigraph of f at the point $(x, f(x))$. Since their introduction in [11], sublinear mappings have been studied by many authors as generalizations of linear transformations, and they possess an extensive theory resembling linear algebra.

It is now commonly understood that the most fruitful approach to the differential analysis of real-valued functions is the "epigraphical" one, based on forming tangents and normals to the epigraph of a function [15]. The information thus obtained is usually represented in the form of subderivatives (closures of directional derivative functions in the convex case) and subgradients, instead of sublinear mappings. The validity of this kind of 'reduction' is tied to the real-valuedness of the function. This paper extends the epigraphical approach to the differential analysis of vector-valued convex functions. All that is needed is a sensible definition of an epigraph, which for K-convex functions turns out to be the K-epigraph. With this choice, the analysis falls into the realm of convex analysis, and we obtain a natural generalization of the subdifferential calculus of real-valued convex functions. The duality between the directional derivative and the subdifferential of real-valued functions turns out to correspond to the relationship between a sublinear mapping and its adjoint [12], and the calculus rules for compound functions correspond to operations defined for sublinear mappings.

A related approach was taken in [6], where the derivative of a vector-valued (not necessarily convex) function was defined to be a particular kind of set-valued mapping called a *fan.* Fans may be also regarded as generalizations of linear transformations, and in some sense they resemble sublinear mappings. However, some properties of sublinear mappings which are not possessed by fans turn out to be crucial in the study of K-convex functions.

Our approach is intimately related to the graphical differentation of general set-valued mappings [1, 2, 15], and especially to *coderivatives*, which correspond to normal cones of mapping graphs [10]. In [15, Example 8.35], Rockafellar and Wets actually define a sublinear mapping which is identical to the generalized Jacobian, to be defined in the Section 4, in the special case of real-valued functions. The calculus rules obtained for coderivatives of general mappings rely on local constraint qualifications; see for example [15, Theorem 10.37]. Our derivation of the calculus rules for K-convex functions also relies on the normal cones, but the ideas employed come from convex analysis. This approach leads to generally sharper results and to global constraint qualifications in the spirit of subdifferential calculus of real-valued convex functions.

We should emphasize, that the definition of an epigraph for a vector-valued function is not new; see for example [13, page 47]. The new idea is to use K-epigraphs as the starting point for the differential analysis of vector-valued convex functions.

In the next section, we briefly review the theory of sublinear mappings. In Section 3, we study the preservation of the convexity of vector-valued functions under sum and composition operations. Section 4 introduces the generalized Jacobian of a K-convex function, relating it to the Jacobian in classical analysis and to the subdifferential of real-valued convex functions. Section 5 gives rules for calculating generalized Jacobians for the sums and compositions of K-convex functions. Section 6 closes the paper by applying the theory of K-Jacobians to a generalized form of convex programming.

2 Sublinear mappings

A *sublinear mapping* [12, Section 39] from X to U is a set-valued mapping whose graph is a convex cone in $X \times U$ containing the origin. Equivalently, a set-valued mapping A from X to U is sublinear if and only if

(a) $Ax_1 + Ax_2 \subset A(x_1 + x_2),\ \forall x_1, x_2 \in X,$

(b) $A(\lambda x) = \lambda Ax,\ \forall x \in X,\ \lambda > 0,$

(c) $0 \in A0.$

The *domain* dom A and *range* ran$A$ of a sublinear mapping A are defined by

$$\begin{aligned} \operatorname{dom} A &= \{x \mid Ax \neq \emptyset\} \\ \operatorname{ran} A &= \{u \mid \exists x \text{ s.t. } u \in Ax\}. \end{aligned}$$

Clearly, all linear transformations (interpreted as mappings whose values are sets of a single point) are sublinear mappings. Conversely, by [12, Theorem 39.1], a single-valued sublinear mapping whose domain is the whole space must be a linear transformation.

The *sum* of two sublinear mappings A_1 and A_2 from X to U is a set-valued mapping from X to U defined by

$$(A_1 + A_2)x = A_1x + A_2x = \{u_1 + u_2 \mid u_1 \in A_1x,\ u_2 \in A_2x\}.$$

The *product* BA of a sublinear mapping A from X to U and a sublinear mapping B from U to V is a set-valued mapping from X to V defined by

$$(BA)x = \{y \mid \exists u \in Ax \text{ s.t. } y \in Bu\}.$$

For a sublinear mapping A from X to U and a convex set $C \subset X$, the *image of C under A* is defined by

$$AC = \{u \mid \exists x \in C \text{ s.t. } u \in Ax\}$$

It is a simple matter to check that $A_1 + A_2$ and BA are again sublinear mappings, and the set AC is convex [12, Section 39]. These properties will be crucial in deriving calculus rules for generalized Jacobians in Section 5.

A sublinear mapping is said to be *closed* if its graph is closed. The closure of the graph $\mathrm{Gr}(A)$ of a sublinear mapping A is a convex cone in $X \times U$ containing the origin, and hence it is the graph of another sublinear mapping. This sublinear mapping is called the *closure* of A and it is denoted by $\mathrm{cl}\, A$.

Given a convex cone $K \subset U$, its *polar cone* K^* is the closed convex cone in U^* defined by

$$K^* = \{u^* \mid \langle u, u^* \rangle \leq 0 \ \forall u \in K\}.$$

We have $(K^*)^* = \mathrm{cl}\, K$ [12, p 121]. For a sublinear mapping A from X to U, we define the *adjoint* to be the sublinear mapping A^* from U^* to X^* with the graph

$$\mathrm{Gr}(A^*) = \{(u^*, x^*) \mid (x^*, -u^*) \in \mathrm{Gr}(A)^*\}.$$

Also, we define the adjoint of a sublinear mapping B from U^* to X^* to be the sublinear mapping B^* from X to U with the graph

$$\mathrm{Gr}(B^*) = \{(x, u) \mid (-u, x) \in \mathrm{Gr}(B)^*\}.$$

Hence, A^* is always a closed and $(A^*)^* = \mathrm{cl}\, A$. If A is a linear transformation, we have

$$\begin{aligned} \mathrm{Gr}(A^*) &= \{(u^*, x^*) \mid \langle x^*, x \rangle + \langle -u^*, Ax \rangle \leq 0 \ \forall x\} \\ &= \{(u^*, x^*) \mid \langle x^* - A^* u^*, x \rangle \leq 0 \ \forall x\} = \{(u^*, x^*) \mid x^* = A^* u^*\}, \end{aligned}$$

which coincides with the definition of adjoint linear transformation.

The following theorems generalize the familiar relations for the adjoints of linear transformations:

Theorem 1 ([12, Theorem 39.5]) *Let A_1 and A_2 be sublinear mappings from X to U. If* $\mathrm{ri}\,\mathrm{dom}\, A_1 \cap \mathrm{ri}\,\mathrm{dom}\, A_2 \neq \emptyset$*, then* $(A_1 + A_2)^* = A_1^* + A_2^*$*. If A_1 and A_2 are closed and* $\mathrm{ri}\,\mathrm{dom}\, A_1^* \cap \mathrm{ri}\,\mathrm{dom}\, A_2^* \neq \emptyset$*, then $A_1 + A_2$ is closed and* $(A_1 + A_2)^* = \mathrm{cl}(A_1^* + A_2^*)$*.*

Theorem 2 ([12, Theorem 39.8]) *Let A be a sublinear mapping from X to U and B a sublinear mapping from U to V. If* $\mathrm{ri}\,\mathrm{ran}\, A \cap \mathrm{ri}\,\mathrm{dom}\, B \neq \emptyset$*, then* $(BA)^* = A^* B^*$*. If A and B are closed and* $\mathrm{ri}\,\mathrm{ran}\, B^* \cap \mathrm{ri}\,\mathrm{dom}\, A^* \neq \emptyset$*, then BA is closed and* $(BA)^* = \mathrm{cl}(A^* B^*)$*.*

3 Convexity of vector-valued functions

For simplicity, we always assume that a cone contains the origin. It follows that the K-convexity of a function $f : X \to U$ defined as the convexity of the K-epigraph

$$\mathrm{epi}_K f = \{(x, u) \mid x \in \mathrm{dom}\, f, \ f(x) - u \in K\}$$

is equivalent to the condition

$$f(\lambda x_1 + (1 - \lambda) x_2) - \lambda f(x_1) - (1 - \lambda) f(x_2) \in K \ \forall x_1, x_2 \in \mathrm{dom}\, f, \ \lambda \in (0, 1).$$

By the definition of a polar cone, the function $\langle u^*, f\rangle$ defined by $\langle u^*, f\rangle(x) = \langle u^*, f(x)\rangle$ is a real-valued convex function with domain $\operatorname{dom} f$ for every $u^* \in K^*$ if and only if f is $(\operatorname{cl} K)$-convex. When the cone K is pointed, *i.e.* not containing any lines, there is a one-to-one correspondence between the pairs (f, K) and epigraphs of functions generated by cones. That is, for a given $\operatorname{epi}_K f$ with K pointed, one can always find the value of f at a given point x, just by inspecting the epigraph. However, the analysis below does not generally assume pointedness of K.

One can always extend the domain of definition of a real-valued function to be all of X by allowing the function to take values in the extended real line $[-\infty, \infty]$. For vector-valued functions, this kind of extension is less natural, and therefore we prefer to explicitly state the domain on which a function has well-defined values in U. The fact that the domain $\operatorname{dom} f$ of a K-convex function $f : X \to U$ is the projection of $\operatorname{epi}_K f$ to X suggests the following definition: the *K-range* $\operatorname{ran}_K f$ of a K-convex function $f : X \to U$ is the projection of $\operatorname{epi}_K f$ to U, *i.e.*

$$\operatorname{ran}_K f = \bigcup_{x \,\in\, \operatorname{dom} f} (f(x) - K) = \left(\bigcup_{x \,\in\, \operatorname{dom} f} f(x) \right) - K.$$

As a projection of a convex set, $\operatorname{ran}_K f$ is convex.

Clearly, $\operatorname{dom} f$ is convex if f is convex with respect to some convex cone. Conversely, every function with a convex domain and values in U is U-convex. We say that a function is *affine* if it is $\{0\}$-convex, or equivalently, convex with respect to every cone in U containing the origin. One could also define a function to be K-concave if $-f$ is K-convex, but such a definition is unnecessary, since K-concavity can be expressed as $(-K)$-convexity.

In the one-dimensional space $\Re$, there are only four cones containing the origin, namely $\{0\}$, $\Re$ itself, and the nonnegative and nonpositive reals, denoted by $\Re_+$ and $\Re_-$, respectively. If f is a real-valued function, then $\Re_-$- and $\Re_+$-convexity mean convexity and concavity, respectively, of f in the usual sense, whereas $\{0\}$-convexity means that f is affine, and $\Re$-convexity means that $\operatorname{dom} f$ is convex.

Given a function f, what are the cones K with respect to which f is K-convex? The smaller the K, the stronger the property K-convexity is, in the sense that if K_1 and K_2 are convex cones containing the origin such that $K_1 \subset K_2$, then K_1-convexity implies K_2-convexity. The smallest closed convex cone K_f with respect to which f is convex is uniquely determined by

$$\begin{aligned} K_f^* &= \{u^* \in U^* \mid \langle u^*, f\rangle \text{ is convex } \} \\ &= \{u^* \in U^* \mid \langle u^*, f\rangle (\lambda x_1 + (1-\lambda)x_2) \le \lambda \langle u^*, f\rangle (x_1) + (1-\lambda) \langle u^*, f\rangle (x_2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \lambda \in (0,1),\ x_1, x_2 \in \operatorname{dom} f\}. \end{aligned}$$

Note that, as an intersection of homogeneous closed inequalities, the right hand side is a closed convex cone. Every function f with a convex domain is thus K_f-convex, and to say that f is K-convex for a closed K means that $K_f \subset K$.

Vector-valued convex functions may be combined in various ways to obtain new vector-valued convex functions. We will consider the operations of addition and composition. When specialized to linear transformations, these operations reduce to the usual sum and product.

For two functions f_1 and f_2 from X to U, the sum f_1+f_2 of f_1 and f_2 is defined in the obvious way by

$$\operatorname{dom}(f_1+f_2)=\operatorname{dom} f_1 \cap \operatorname{dom} f_2$$

and

$$(f_1+f_2)(x)=f_1(x)+f_2(x),\ \forall x\in\operatorname{dom}(f_1+f_2).$$

Theorem 3 *If f_1 and f_2 are functions from X to U, K_1-convex and K_2-convex respectively, then f_1+f_2 is a (K_1+K_2)-convex function from X to U.*

Proof. For any $x_1, x_2 \in \operatorname{dom}(f_1+f_2)$ and $i=1,2$, we have

$$f_i(\lambda x_1+(1-\lambda)x_2)-\lambda f_i(x_1)-(1-\lambda)f_i(x_2)\in K_i\ \forall\lambda\in(0,1).$$

Adding, we obtain

$$(f_1+f_2)(\lambda x_1+(1-\lambda)x_2)-\lambda(f_1+f_2)(x_1)-(1-\lambda)(f_1+f_2)(x_2)\in K_1+K_2\ \forall\lambda\in(0,1).$$

☐

Corollary 1 *If f_1 is a K_1-convex function from X to U_1 and f_2 is a K_2-convex function from X to U_2, then the function $(f_1,f_2): X\to U_1\times U_2$ defined by $\operatorname{dom}(f_1,f_2)=\operatorname{dom} f_1\cap\operatorname{dom} f_2$ and $(f_1,f_2)(x)=(f_1(x),f_2(x))$ is $K_1\times K_2$-convex.*

Proof. Apply the theorem to functions $\tilde{f}_1(x)=(f_1(x),0)$ and $\tilde{f}_2(x)=(0,f_2(x))$ from X to $U_1\times U_2$, which are $K_1\times\{0\}$-convex and $\{0\}\times K_2$-convex respectively. ☐

The *recession cone* $\operatorname{rc} C$ of a convex set C is the set of vectors y such that for any $x\in C$, $x+\lambda y\in C$ for every $\lambda\geq 0$. The *recession cone* of a real-valued convex function f is the set of vectors y such that $(y,0)\in\operatorname{rc}\operatorname{epi} f$ or equivalently, such that for any $x\in\operatorname{dom} f$

$$f(x+\lambda y)\leq f(x)\ \forall\lambda\geq 0.$$

This set, which we denote by $\operatorname{rc} f$, gives the set of directions in which f is nonincreasing. Accordingly, we define the *K-recession cone* $\operatorname{rc}_K f$ of a K-convex function f by

$$\begin{aligned}\operatorname{rc}_K f &= \{y \mid (y,0)\in\operatorname{rc}\operatorname{epi}_K f\}\\ &= \{y \mid f(x+\lambda y)-f(x)\in K\ \forall\lambda\geq 0,\ \forall x\in\operatorname{dom} f\}.\end{aligned}$$

For a function $h: X\to U$ and a function $g: U\to V$, the *composition* $g\circ h$ of h and g is a function from X to V defined by

$$\operatorname{dom}(g\circ h)=h^{-1}(\operatorname{dom} g)=\{x\in\operatorname{dom} h \mid h(x)\in\operatorname{dom} g\},$$

and

$$(g\circ h)(x)=g(h(x))\ \forall x\in\operatorname{dom}(g\circ h).$$

Theorem 4 *Let* $h : X \to U$ *be* K_U*-convex, and let* $g : U \to V$ *be* K_V*-convex. If* $K_U \subset \mathrm{rc}_{K_V} g$*, then* $g \circ h : X \to V$ *is* K_V*-convex.*

Proof. Let $x_1, x_2 \in \mathrm{dom}(g \circ h)$ and $\lambda \in (0,1)$. By the K_U-convexity of h

$$h(\lambda x_1 + (1-\lambda)x_2) - \lambda h(x_1) - (1-\lambda)h(x_2) \in K_U,$$

so that because $K_U \subset \mathrm{rc}_{K_V} g$,

$$g(h(\lambda x_1 + (1-\lambda)x_2)) - g(\lambda h(x_1) + (1-\lambda)h(x_2)) \in K_V.$$

Using K_V-convexity of g we have

$$g(\lambda h(x_1) + (1-\lambda)h(x_2)) - \lambda g(h(x_1)) - (1-\lambda)g(h(x_2)) \in K_V.$$

Adding up, we obtain

$$g(h(\lambda x_1 + (1-\lambda)x_2)) - \lambda g(h(x_1)) - (1-\lambda)g(h(x_2)) \in K_V.$$

□

Corollary 2 *If* A *is a linear transformation from* X *to* U*, and* g *a* K_V*-convex function from* U *to* V*, then the function* gA *from* X *to* V *defined by*

$$(gA)(x) = g(Ax),$$

is K_V*-convex with domain* $A^{-1}\mathrm{dom}\, g$*.*

If h *is a* K_U*-convex function from* X *to* U*, and* B *is a linear transformation from* U *to* V*, then the function* Bh *from* X *to* V *defined by*

$$(Bh)(x) = Bh(x),$$

is (BK_U)*-convex with domain* $\mathrm{dom}\, h$*.*

Proof. For the first part, choose $h = A$ and $K_U = \{0\}$. The origin belongs to the recession cone of any function, and the polar cone of $\{0\} \subset U$ is U^*, so that the assumptions of the theorem are satisfied. For the second part, choose $g = B$ and $K_V = BK_U$. We have $\mathrm{rc}_{(BK_U)}\, g = \{u \mid Bu \in BK_U\} = B^{-1}BK_U$, so that $K_U \subset \mathrm{rc}_{(BK_U)}\, g$, which implies the (BK_U)-convexity. □

The *indicator function* δ_C of a convex set C is defined by

$$\delta_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ +\infty & \text{if } u \notin C. \end{cases}$$

The following theorem will be useful in expressing $\mathrm{ri}\,\mathrm{dom}(g \circ h)$ in terms of $\mathrm{dom}\, h$ and $\mathrm{dom}\, g$.

Theorem 5 *Let h be a K-convex function from X to U and C a convex set in U such that $K \subset \operatorname{rc} C$. Then the set*

$$h^{-1}(C) = \{x \mid h(x) \in C\}$$

is convex, and if $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} C \neq \emptyset$, *then*

$$\operatorname{ri} h^{-1}(C) = \{x \in \operatorname{ri} \operatorname{dom} h \mid h(x) \in \operatorname{ri} C\}.$$

Moreover, the condition $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} C \neq \emptyset$ *holds if and only if there exists an* $x \in \operatorname{ri} \operatorname{dom} h$ *such that* $h(x) \in \operatorname{ri} C$.

Proof. The convexity follows from the previous theorem by choosing $g = \delta_C$. Because $0 \in K$ and $K \subset \operatorname{rc} C$, we have $C = C + K$, and thus

$$\begin{aligned} h^{-1}(C) &= \{x \in \operatorname{dom} h \mid (h(x) - K) \cap C \neq \emptyset\} \\ &= P_X(\operatorname{epi}_K h \cap (X \times C)), \end{aligned}$$

where P_X denotes the linear transformation $(x, u) \mapsto x$. Because by [12, Theorem 6.6], $\operatorname{ri} \operatorname{ran}_K h = P_U \operatorname{ri} \operatorname{epi}_K f$, the condition $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} C \neq \emptyset$ implies that $\operatorname{ri} \operatorname{epi}_K h \cap \operatorname{ri}(X \times C) \neq \emptyset$, and thus by Theorems 6.6 and 6.5 of [12] we have

$$\operatorname{ri} h^{-1}(C) = P_X(\operatorname{ri} \operatorname{epi}_K h \cap \operatorname{ri}(X \times C)).$$

By [12, Theorem 6.8]

$$\operatorname{ri} \operatorname{epi}_K h = \{(x, u) \mid x \in \operatorname{ri} \operatorname{dom} h,\ u \in \operatorname{ri}(h(x) - K)\},$$

so that, because $\operatorname{ri}(X \times C) = X \times \operatorname{ri} C$,

$$\operatorname{ri} h^{-1}(C) = \{x \in \operatorname{ri} \operatorname{dom} h \mid \operatorname{ri}(h(x) - K) \cap \operatorname{ri} C \neq \emptyset\},$$

where by [12, Corollary 6.6.2], $\operatorname{ri}(h(x) - K) \cap \operatorname{ri} C \neq \emptyset$ may be written as $h(x) \in \operatorname{ri} C + \operatorname{ri} K = \operatorname{ri}(C + K) = \operatorname{ri} C$.

By [12, Theorem 6.6], we have

$$\operatorname{ri} \operatorname{ran}_K h = \bigcup_{x \in \operatorname{ri} \operatorname{dom} h} \operatorname{ri}(h(x) - K),$$

and thus, $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} C \neq \emptyset$ may be written as

$$\left[\bigcup_{x \in \operatorname{ri} \operatorname{dom} h} \operatorname{ri}(h(x) - K)\right] \cap \operatorname{ri} C \neq \emptyset,$$

or

$$\bigcup_{x \in \operatorname{ri} \operatorname{dom} h} [\operatorname{ri}(h(x) - K) \cap \operatorname{ri} C] \neq \emptyset,$$

which means that there exists an $x \in \operatorname{ri} \operatorname{dom} h$ such that $\operatorname{ri}(h(x) - K) \cap \operatorname{ri} C \neq \emptyset$. As already noted, the last condition may be written as $h(x) \in \operatorname{ri} C$. $\square$

4 Generalized Jacobian of a K-convex function

The *tangent cone* of a convex set $C \subset X$ at a point $\bar{x} \in C$ is defined to be the closure of the set of all vectors x such that $\bar{x} + \epsilon x \in C$ for some $\epsilon > 0$. This is the closure of the convex cone generated by $C - \bar{x}$:

$$T_C(\bar{x}) = \operatorname{cl}\{\lambda x \mid \lambda > 0,\ x \in C - \bar{x}\}.$$

The *normal cone* of C at $\bar{x}$ is the set

$$N_C(\bar{x}) = \{x^* \mid \langle x^*, x - \bar{x}\rangle \le 0\ \forall x \in C\}.$$

It is easily shown that the tangent and normal cones at a point $\bar{x}$ are polar to one another. The tangent cone $T_C(\bar{x})$ gives a local approximation of the set C at $\bar{x}$, and the normal cone is just a dual characterization of this approximation, containing exactly the same information.

A tangent cone at a boundary point $(\bar{x}, f(\bar{x}))$ of the K-epigraph of a K-convex function f gives a local approximation of the boundary of the epigraph, which may be identified with the pair (f, K) when K is pointed. In any case, $T_{\mathrm{epi}_K f}(\bar{x}, f(\bar{x}))$ is a certain closed convex cone in $X \times U$, and hence it corresponds to a closed sublinear mapping from X to U. For a K-convex function f from X to U, we define the *K-Jacobian of f at* $\bar{x} \in \operatorname{dom} f$, denoted by $\mathrm{D}_K f(\bar{x})$, to be the closed sublinear mapping from X to U with the graph

$$\mathrm{Gr}(\mathrm{D}_K f(\bar{x})) = T_{\mathrm{epi}_K f}(\bar{x}, f(\bar{x})).$$

The sublinear-mapping-valued mapping $\mathrm{D}_K f : x \to \mathrm{D}_K f(x)$ is called the *K-Jacobian of f.*

If the limit

$$f'(\bar{x}; x) = \lim_{\mu \downarrow 0} \frac{f(\bar{x} + \mu x) - f(\bar{x})}{\mu}$$

exists, it is called the *directional derivative of f at $\bar{x}$ in direction x.* If $f'(\bar{x}; x)$ exists for every x, then $f'(\bar{x}; \cdot)$ is a well-defined positively homogeneous function from X to U, with X as its domain.

Theorem 6 *Let f be a K-convex function from X to U, and $\bar{x} \in \operatorname{dom} f$ such that $f'(\bar{x}; x)$ exists for all $x \in X$. If K is closed, then*

$$\operatorname{cl} \mathrm{epi}_K f'(\bar{x}; \cdot) = \mathrm{Gr}(\mathrm{D}_K f(\bar{x})).$$

Proof. By definition,

$$\begin{aligned}
\mathrm{Gr}(\mathrm{D}_K f(\bar{x})) &= \operatorname{cl}\{\lambda(x,u) \mid (x,u) \in \mathrm{epi}_K f - (\bar{x}, f(\bar{x})),\ \lambda > 0\} \\
&= \operatorname{cl}\{\lambda(x,u) \mid f(\bar{x} + x) - u - f(\bar{x}) \in K,\ \lambda > 0\} \\
&= \operatorname{cl}\{(x,u) \mid \exists \mu > 0 \text{ s.t. } f(\bar{x} + \mu x) - \mu u - f(\bar{x}) \in K\} \\
&= \operatorname{cl}\left\{(x,u) \;\middle|\; \exists \mu > 0 \text{ s.t. } \frac{1}{\mu}(f(\bar{x} + \mu x) - f(\bar{x})) - u \in K\right\}.
\end{aligned}$$

If $\mu_1 \geq \mu_2 > 0$ and $\bar{x} + \mu_1 x \in \operatorname{dom} f$, we have by K-convexity of f that

$$f(\lambda x_1 + (1-\lambda)x_2) - \lambda f(x_1) - (1-\lambda) f(x_2) \in K,$$

where $x_1 = \bar{x}$, $x_2 = \bar{x} + \mu_1 x$ and $\lambda = 1 - \mu_2/\mu_1$. Dividing by μ_2, this may be written as

$$\frac{f(\bar{x} + \mu_2 x) - f(\bar{x})}{\mu_2} - \frac{f(\bar{x} + \mu_1 x) - f(\bar{x})}{\mu_1} \in K.$$

Thus, assuming that the limit

$$f'(\bar{x}; x) = \lim_{\mu \downarrow 0} \frac{f(\bar{x} + \mu x) - f(\bar{x})}{\mu}$$

exists for every x, we have by the closedness of K that

$$\operatorname{Gr}(\mathrm{D}_K f(\bar{x})) \subset \operatorname{cl} \{(x,u) \mid f'(\bar{x}; x) - u \in K\} = \operatorname{cl} \operatorname{epi}_K f'(\bar{x}; \cdot).$$

On the other hand, if $(x,u) \in \operatorname{epi}_K f'(\bar{x}; x)$, we have for any $\mu > 0$ that

$$\frac{f(\bar{x} + \mu x) - f(\bar{x}))}{\mu} - u(\mu) \in K,$$

where

$$u(\mu) = u + \frac{f(\bar{x} + \mu x) - f(\bar{x}))}{\mu} - f'(\bar{x}; x).$$

We then have that $(\bar{x} + \mu x, f(\bar{x}) + \mu u(\mu)) \in \operatorname{epi}_K f$, and thus $(x, u(\mu)) \in \operatorname{Gr}(\mathrm{D}_K f(\bar{x}))$. Letting $\mu \downarrow 0$, we see that $u(\mu) \to u$, which by closedness of $\operatorname{Gr}(\mathrm{D}_K f(\bar{x}))$, implies that $(x,u) \in \operatorname{Gr}(\mathrm{D}_K f(\bar{x}))$. By the arbitrary choice of (x,u), we have $\operatorname{epi}_K f'(\bar{x}; \cdot) \subset \operatorname{Gr}(\mathrm{D}_K f(\bar{x}))$, and again using the closedness of $\operatorname{Gr}(\mathrm{D}_K f(\bar{x}))$, we have the reverse inclusion

$$\operatorname{cl} \operatorname{epi}_K f'(\bar{x}; \cdot) \subset \operatorname{Gr}(\mathrm{D}_K f(\bar{x})).$$

□

If f is real-valued and $K = \Re_-$, then by Theorem 23.1 of [12], $f'(\bar{x}; \cdot)$ is well defined as an extended real-valued function for any $\bar{x} \in \operatorname{dom} f$, and the proof of the above theorem shows that the closure of its epigraph equals the graph of the $\Re_-$-Jacobian. By Propositions 3.4a and 3.7 of [3], the directional derivative is well defined for vector-valued K-convex functions whenever $\bar{x} \in \operatorname{int} \operatorname{dom} f$ and K is closed and pointed, but in general $f'(\bar{x}; \cdot)$ may not exist.

By contrast, for any K and K-convex function f, the K-Jacobian is well defined throughout $\operatorname{dom} f$. We have the following result relating the domain and K-range of a K-convex function to the domain and range of its K-Jacobian at a given point:

Lemma 1 *Let f be a K-convex function and $\bar{x} \in \operatorname{dom} f$. Then*

$$\begin{aligned} \operatorname{cl} \operatorname{dom} \mathrm{D}_K f(\bar{x}) &= T_{\operatorname{dom} f}(\bar{x}) \\ \operatorname{cl} \operatorname{ran} \mathrm{D}_K f(\bar{x}) &= T_{\operatorname{ran}_K f}(f(\bar{x})). \end{aligned}$$

Proof. By Proposition 4.2.9 of [2], $\mathrm{cl}(AT_C(z)) = T_{AC}(Az)$ for any linear transformation A and convex set C. Setting $C = \mathrm{epi}_K f$ and $z = (\bar{x}, f(\bar{x}))$, we obtain the formulas by choosing A to be the projections to X and U, respectively. ☐

This shows, in particular, that if $\bar{x} \in \mathrm{ri}\,\mathrm{dom}\, f$, $\mathrm{dom}(\mathrm{D}_K f(\bar{x}))$ is the subspace parallel to the affine hull of $\mathrm{dom}\, f$, and if $\bar{x} \in \mathrm{int}\,\mathrm{dom}\, f$, $\mathrm{dom}(\mathrm{D}_K f(\bar{x}))$ is all of X.

The next theorem gives an expression for the adjoint of the K-Jacobian. We will use the notation

$$\mathrm{D}_K^* f(x) = [\mathrm{D}_K f(x)]^*.$$

Theorem 7 *Let f be a K-convex function and $\bar{x} \in \mathrm{dom}\, f$. Then the adjoint of the K-Jacobian of f at $\bar{x}$ is the closed sublinear mapping from U to X given by*

$$\mathrm{D}_K^* f(\bar{x})u^* = \begin{cases} \partial\langle u^*, f\rangle(\bar{x}) & \text{if } u^* \in K^* \\ \emptyset & \text{if } u^* \notin K^*. \end{cases}$$

Proof. Because

$$[\mathrm{Gr}(\mathrm{D}_K f(\bar{x}))]^* = [T_{\mathrm{epi}_K f}(\bar{x}, f(\bar{x}))]^* = N_{\mathrm{epi}_K f}(\bar{x}, f(\bar{x})),$$

we have that $\mathrm{Gr}(\mathrm{D}_K^* f(\bar{x}))$ is equal to

$$\begin{aligned}
& \{(u^*, x^*) \mid (x^*, -u^*) \in [\mathrm{Gr}(\mathrm{D}_K f(\bar{x}))]^*\} \\
= \; & \{(u^*, x^*) \mid (x^*, -u^*) \in N_{\mathrm{epi}_K f}(\bar{x}, f(\bar{x}))\} \\
= \; & \{(u^*, x^*) \mid \langle x - \bar{x}, x^*\rangle + \langle f(\bar{x}) - u, u^*\rangle \le 0 \ \forall (x, u) \in \mathrm{epi}_K f\} \\
= \; & \{(u^*, x^*) \mid \langle x - \bar{x}, x^*\rangle + \langle f(\bar{x}) - u, u^*\rangle \le 0 \ \forall x \in \mathrm{dom}\, f,\ u \in f(x) - K\} \\
= \; & \{(u^*, x^*) \mid \langle x - \bar{x}, x^*\rangle + \langle f(\bar{x}) - f(x), u^*\rangle + \langle v, u^*\rangle \le 0 \ \forall x \in \mathrm{dom}\, f,\ v \in K\} \\
= \; & \{(u^*, x^*) \mid u^* \in K^*,\ \langle x - \bar{x}, x^*\rangle + \langle f(\bar{x}) - f(x), u^*\rangle \le 0 \ \forall x \in \mathrm{dom}\, f\}.
\end{aligned}$$

Hence, for $u^* \notin K^*$, $\mathrm{D}_K^* f(\bar{x})u^* = \emptyset$, and for $u^* \in K^*$

$$\mathrm{D}_K^* f(\bar{x})u^* = \{x^* \mid \langle u^*, f(x)\rangle \ge \langle u^*, f(\bar{x})\rangle + \langle x^*, x - \bar{x}\rangle \ \forall x \in \mathrm{dom}\, f\}.$$

☐

This result relates the generalized Jacobian to the subdifferential operator of real-valued convex functions, and it gives a way to infer properties of one from the properties of the other. In particular, when f is real-valued and $K = \Re_-$, the graph of $\mathrm{D}_K^* f(\bar{x})$ is the closed convex cone generated by the set $\{(1, x^*) \mid x^* \in \partial f(\bar{x})\}$, and $\partial f(\bar{x}) = \mathrm{D}_K^* f(\bar{x})1$.

According to the above theorem, we always have $\mathrm{dom}\, \mathrm{D}_K^* f(x) \subset K^*$, but, in general, this does not have to hold as an equality. The first part of the next theorem gives a sufficient condition for the equality to hold.

Theorem 8 *For any K-convex function f and $x \in \operatorname{ri} \operatorname{dom} f$, we have $\operatorname{dom}(\mathrm{D}_K^* f(x)) = K^*$. Moreover, for any $x \in \operatorname{dom} f$ and $u^* \in \operatorname{dom} \mathrm{D}_K^* f(x)$, $\operatorname{rc}(\mathrm{D}_K^* f(x)u^*) = N_{\operatorname{dom} f}(x)$. In particular, $\mathrm{D}_K^* f(x)u^*$ is a nonempty bounded set for all $u^* \in K^*$ if and only if $x \in \operatorname{int} \operatorname{dom} f$.*

Proof. For any $u^* \in K^*$, $\operatorname{dom}\langle u^*, f\rangle = \operatorname{dom} f$, so that by [12, Theorem 23.4], $\mathrm{D}_K^* f(x)u^* = \partial\langle u^*, f\rangle \neq \emptyset$ whenever $x \in \operatorname{ri} \operatorname{dom} f$. As shown on page 415 of [12], $\operatorname{rc} Ax = \operatorname{rc} A0$ for any closed sublinear mapping A and for any $x \in \operatorname{dom} A$. Since the adjoint of the K-Jacobian is closed, we have by Theorem 7 for all $u^* \in \operatorname{dom} \mathrm{D}_K^* f(x)$ that

$$\operatorname{rc}(\mathrm{D}_K^* f(x)u^*) = \operatorname{rc}(\mathrm{D}_K^* f(x)0) = \{x^* \mid 0 \geq \langle x^*, x - \bar{x}\rangle \; \forall x \in \operatorname{dom} f\} = N_{\operatorname{dom} f}(x).$$

By [12, Theorem 8.4], a closed convex set is bounded if and only if its recession cone is $\{0\}$. The last statement follows by applying this result to the sets $\mathrm{D}_K^* f(x)u^*$. □

The following theorem relates $\mathrm{D}_K f$ to the classical Jacobian ∇f, and it generalizes [12, Theorem 25.1] on the differentiability of real-valued convex functions.

Theorem 9 *Let f be a K-convex function from X to U and $\bar{x} \in \operatorname{dom} f$. If f is differentiable at $\bar{x}$, then*

$$\mathrm{D}_K f(\bar{x})x = \nabla f(\bar{x})x - K \; \forall x \in X$$

and

$$\mathrm{D}_K^* f(\bar{x})u^* = [\nabla f(\bar{x})]^* u^* \; \forall u^* \in K^*.$$

If K is pointed, we have the converse implication: if $\mathrm{D}_K^* f(\bar{x})u^*$ is single-valued for any $u^* \in \operatorname{int} K^*$, then f is differentiable at $\bar{x}$.

Proof. When f is differentiable at $\bar{x}$, we have $f'(\bar{x}; \cdot) = \nabla f(\bar{x})$, and the first equality follows from Theorem 6. The expression for the adjoint is obtained by finding the adjoint of $\mathrm{D}_K f(\bar{x})$, or by noting that the differentiability of f implies that of $\langle u^*, f\rangle$, so that $\mathrm{D}_K^* f(\bar{x})u^* = \nabla \langle u^*, f\rangle (\bar{x}) = [\nabla f(\bar{x})]^* u^*$ for all $u^* \in K^*$.

If $\mathrm{D}_K^* f(\bar{x})u^*$ is single-valued, $\langle u^*, f\rangle$ is differentiable and $\bar{x} \in \operatorname{int} \operatorname{dom} \langle u^*, f\rangle = \operatorname{int} \operatorname{dom} f$, by Theorem 25.1 and Corollary 25.1.1 of [12]. By [12, Theorem 13.1], $u^* \in \operatorname{int} K^*$ means that $\langle u, u^*\rangle < 0 \; \forall u \in K \setminus \{0\}$. Thus, when K is pointed, $f'(\bar{x}; \cdot)$ is well-defined and linear by Propositions 3.7 and 4.2 of [3]. By [17, Corollary 3.2], $\bar{x} \in \operatorname{int} \operatorname{dom} f$ implies that f is continuous at $\bar{x}$, and thus by [3, Corollary 2.4d], f is Locally Lipschitz near $\bar{x}$, which together with linearity of $f'(\bar{x}; \cdot)$ implies the differentiability of f at $\bar{x}$. □

5 Calculus rules for generalized Jacobians

The usefulness of any concept of a differential hinges on the ease of computing it for a given function. One of the most serious shortcomings of the subdifferential calculus of convex analysis is the lack of a general chain rule of subdifferentation. Using Theorem 7 and the theory of subdifferentials, we are able to derive more complete calculus rules for the generalized Jacobians defined in the previous section.

The following lemma will be useful.

Lemma 2 *Let* C_1 *and* C_2 *be convex sets. If* $\mathrm{ri}\, C_1 \cap \mathrm{ri}\, C_2 \neq \emptyset$*, then for every* $x \in C_1 \cap C_2$*,*

$$\mathrm{ri}\, T_{C_1}(x) \cap \mathrm{ri}\, T_{C_2}(x) \neq \emptyset.$$

Proof. Let $x \in C_1 \cap C_2$ and define

$$T_i = \{\lambda y \mid \lambda > 0,\ y \in \mathrm{ri}(C_i - x)\} \quad i = 1, 2.$$

Writing the condition $\mathrm{ri}\, C_1 \cap \mathrm{ri}\, C_2 \neq \emptyset$ in the equivalent form $\mathrm{ri}\,(C_1 - x) \cap \mathrm{ri}\,(C_2 - x) \neq \emptyset$, we see that $T_1 \cap T_2 \neq \emptyset$. The rest is implied by the inclusion $T_i \subset \mathrm{ri}\, T_{C_i}(x)$. □

We begin by the formula for the Jacobian of the sum of K-convex functions. For two sublinear mappings A_1 and A_2, we write $A_1 \subset A_2$ if $A_1 x \subset A_2 x\ \forall x$, *i.e.* if $\mathrm{Gr}(A_1) \subset \mathrm{Gr}(A_2)$.

Theorem 10 *If* f_1 *and* f_2 *are functions from* X *to* U*,* K_1*-convex and* K_2*-convex respectively, then for any* $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$

$$\mathrm{D}^*_{K_1+K_2}(f_1 + f_2)(x) \supset \mathrm{D}^*_{K_1} f_1(x) + \mathrm{D}^*_{K_2} f_2(x).$$

If $\mathrm{ri\, dom}\, f_1 \ \cap\ \mathrm{ri\, dom}\, f_2 \neq \emptyset$*, then*

$$\mathrm{D}^*_{K_1+K_2}(f_1 + f_2)(x) = \mathrm{D}^*_{K_1} f_1(x) + \mathrm{D}^*_{K_2} f_2(x)$$

and

$$\mathrm{D}_{K_1+K_2}(f_1 + f_2)(x) = \mathrm{cl}(\mathrm{D}_{K_1} f_1(x) + \mathrm{D}_{K_2} f_2(x)).$$

Proof. By Theorem 3, $f_1 + f_2$ is a $(K_1 + K_2)$-convex function, so that by Theorem 7, we have for each $u^* \in (K_1 + K_2)^* = K_1^* \cap K_2^*$ that

$$\begin{aligned} \mathrm{D}^*_{K_1+K_2}(f_1 + f_2)(x)u^* &= \partial\langle u^*, f_1 + f_2\rangle(x) \\ &= \partial(\langle u^*, f_1\rangle + \langle u^*, f_2\rangle)(x). \end{aligned}$$

Using Theorem 23.8 of [12] on the subdifferential of the sum of real-valued convex functions, we have

$$\begin{aligned} \mathrm{D}^*_{K_1+K_2}(f_1 + f_2)(x)u^* &\supset \partial\langle u^*, f_1\rangle(x) + \partial\langle u^*, f_2\rangle(x) \\ &= \mathrm{D}^*_{K_1} f_1(x)u^* + \mathrm{D}^*_{K_2} f_2(x)u^* \\ &= (\mathrm{D}^*_{K_1} f_1(x) + \mathrm{D}^*_{K_2} f_2(x))u^*, \end{aligned}$$

which holds as an equality under the condition $\mathrm{ri\, dom}\langle u^*, f_1\rangle \ \cap\ \mathrm{ri\, dom}\langle u^*, f_2\rangle \neq \emptyset$. The equality for the adjoint Jacobians follows by noting that $\mathrm{dom}\langle u^*, f_i\rangle = \mathrm{dom}\, f_i$.

By Lemma 1 and Theorem 6.3 of [12], $\mathrm{ri\, dom}\, \mathrm{D}_{K_i} f_i(x) = \mathrm{ri}\, T_{\mathrm{dom}\, f_i}(x)$, so that by Lemma 2, the condition $\mathrm{ri\, dom}\, f_1 \ \cap\ \mathrm{ri\, dom}\, f_2 \neq \emptyset$ implies $\mathrm{ri\, dom}\, \mathrm{D}_{K_1} f_1(x) \cap \mathrm{ri\, dom}\, \mathrm{D}_{K_2} f_2(x) \neq \emptyset$. Hence, by the equality for the adjoint Jacobians and Theorem 1,

$$\begin{aligned} \mathrm{D}_{K_1+K_2}(f_1 + f_2)(x) &= (\mathrm{D}^*_{K_1} f_1(x) + \mathrm{D}^*_{K_2} f_2(x))^* \\ &= (\mathrm{D}_{K_1} f_1(x) + \mathrm{D}_{K_2} f_2(x))^{**} \\ &= \mathrm{cl}(\mathrm{D}_{K_1} f_1(x) + \mathrm{D}_{K_2} f_2(x)). \end{aligned}$$

□

Remark. If $x \in \operatorname{ri}\operatorname{dom} f_1 \cap \operatorname{ri}\operatorname{dom} f_2$, we have by Theorem 8 that $\operatorname{dom} \mathrm{D}^*_K f_i(x) = K^*_i \; i = 1, 2$, so that if $\operatorname{ri} K^*_1 \cap \operatorname{ri} K^*_2 \neq \emptyset$ the closure in the last formula is superfluous by Theorem 1.

The following slight modification of Theorem 24b of [13] is needed in deriving a chain rule for subdifferentation. P_U denotes the projection $(u, x) \mapsto u$.

Lemma 3 *Let F be a convex function on $U \times X$, and define $f(x) = F(0, x)$ and $L(u^*, x) = \inf_u\{\langle u, u^*\rangle + F(u, x)\}$. If $0 \in \operatorname{ri} P_U \operatorname{dom} F$, then*

$$\partial f(x) = \{x^* \mid \exists u^* \text{ s.t. } (0, x^*) \in \partial L(u^*, x)\}.$$

Remark. Here L is a concave-convex saddle function, and $\partial L(u^*, x)$ denotes the set of points (u, x^*) satisfying

$$\begin{array}{rcll} L(v^*, x) & \leq & L(u^*, x) + \langle u, v^* - u^*\rangle & \forall v^* \in U^* \\ L(u^*, y) & \geq & L(u^*, x) + \langle x^*, y - x\rangle & \forall y \in X. \end{array}$$

Proof. According to the Theorem 5.7 of [12], the function $\varphi(u) = \inf_x F(u, x)$ is convex on U. Because $\operatorname{dom} \varphi = P_U \operatorname{dom} F$, the condition $0 \in \operatorname{ri} P_U \operatorname{dom} F$ guarantees by Theorem 23.4 of [12] that φ is subdifferentiable at 0. This implies, by Theorem 16 and Corollary 15A of [13], that $0 \in \partial f(x)$ if and only if there exists a u^* such that $(0, 0) \in \partial L(u^*, x)$. Applying this fact to the convex function $F_{x^*}(u, x) = F(u, x) - \langle x^*, x\rangle$, we see that because $P_U \operatorname{dom} F_{x^*} = P_U \operatorname{dom} F$, $x^* \in \partial f(x)$ if and only if there exists a u^* such that $(0, x^*) \in \partial L(u^*, x)$. □

Theorem 11 *Let h be a K-convex function from X to U and g a real-valued convex function on $\operatorname{dom} g \subset U$ such that $K \subset \operatorname{rc} g$. Then, for any $x \in \operatorname{dom}(g \circ h)$,*

$$\partial(g \circ h)(x) \supset \mathrm{D}^*_K h(x) \partial g(h(x)),$$

which holds as an equality if $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$.

Proof. If $v \in K$ and $u^* \in \partial g(u)$ we have $g(u+v) \geq g(u) + \langle u^*, v\rangle$ and $g(u+v) \leq g(u)$ which implies that $\langle u^*, v\rangle \leq 0$. Thus, $\partial g(u) \subset K^*$ for all u, and by Theorem 7,

$$D^*_K h(x) \partial g(h(x)) = \{x^* \mid \exists u^* \in \partial g(h(x)) \text{ s.t. } x^* \in \partial \left\langle u^*, h\right\rangle(x)\}.$$

$u^* \in \partial g(h(x))$ implies

$$g(h(x')) \geq g(h(x)) + \langle u^*, h(x') - h(x)\rangle \; \forall x' \in \operatorname{dom}(g \circ h),$$

and $x^* \in \partial\langle u^*, h\rangle(x)$ means

$$\langle u^*, h(x')\rangle \geq \langle u^*, h(x)\rangle + \langle x^*, x' - x\rangle.$$

Combining, we have

$$g(h(x')) \geq g(h(x)) + \langle x^*, x' - x\rangle \; \forall x' \in \operatorname{dom}(g \circ h)$$

i.e. $x^* \in \partial(g \circ h)(x)$, which proves the general inclusion.

Now, $(g \circ h)(x) = F(0, x)$, where

$$F(u,x) = \begin{cases} g(h(x)-u) & \text{if } x \in \operatorname{dom} h \\ \infty & \text{if } x \notin \operatorname{dom} h. \end{cases}$$

Because $K \subset \operatorname{rc} g$, we have that $\operatorname{dom} g = \operatorname{dom} g + K$ and

$$\begin{aligned} P_U \operatorname{dom} F &= \{u \mid \exists x \in \operatorname{dom} h \text{ s.t. } h(x) - u \in \operatorname{dom} g\} \\ &= \{u \mid \exists x \in \operatorname{dom} h \text{ s.t. } h(x) - u \in \operatorname{dom} g + K\} \\ &= \{u \mid \exists x \in \operatorname{dom} h \text{ s.t. } u \in h(x) - K - \operatorname{dom} g\} \\ &= \operatorname{ran}_K h - \operatorname{dom} g, \end{aligned}$$

so that by Corollary 6.6.2 of [12],

$$\operatorname{ri} P_U \operatorname{dom} F = \operatorname{ri} \operatorname{ran}_K h \ - \ \operatorname{ri} \operatorname{dom} g.$$

Thus, $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$ implies by the above lemma that

$$\partial(g \circ h)(x) = \{x^* \mid \exists u^* \text{ s.t. } (0, x^*) \in \partial L(u^*, x)\},$$

where

$$\begin{aligned} L(u^*, x) &= \inf_u \{\langle u, u^* \rangle + F(u,x)\} \\ &= \begin{cases} +\infty & \text{if } x \notin \operatorname{dom} h \\ \langle u^*, h(x) \rangle - g^*(u^*) & \text{if } x \in \operatorname{dom} h. \end{cases} \end{aligned}$$

By Theorem 23.5 of [12], $x \in \partial f^*(x^*) \Leftrightarrow x^* \in \partial(\operatorname{cl} f)(x)$ for any proper convex function f, so that

$$\begin{aligned} \partial(g \circ h)(x) &= \{x^* \mid \exists u^* \text{ s.t. } 0 \in h(x) - \partial g^*(u^*), \ x^* \in \partial\langle u^*, h\rangle(x)\} \\ &= \{x^* \mid \exists u^* \in \partial(\operatorname{cl} g)(h(x)) \text{ s.t. } x^* \in \mathrm{D}_K^* h(x) u^*\} \\ &= \mathrm{D}_K^* h(x) \partial(\operatorname{cl} g)(h(x)). \end{aligned}$$

If $\partial g(h(x)) \neq \emptyset$, we have $\partial(\operatorname{cl} g)(h(x)) = \partial g(h(x))$ by Corollary 23.5.2 of [12].

To finish the proof, it suffices to show that $\partial g(h(x)) = \emptyset$ implies $\partial(g \circ h)(x) = \emptyset$. By Theorem 5, the condition $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$ guarantees the existence of an $\bar{x} \in \operatorname{ri} \operatorname{dom} h$ such that $h(\bar{x}) \in \operatorname{ri} \operatorname{dom} g$. By K-convexity of h

$$h(\lambda \bar{x} + (1-\lambda)x) - \lambda h(\bar{x}) - (1-\lambda) h(x) \in K \quad \forall \lambda \in (0,1),$$

so that, because $K \subset \operatorname{rc} g$,

$$g[h(\lambda \bar{x} + (1-\lambda)x)] \leq g[\lambda h(\bar{x}) + (1-\lambda) h(x)] \ \forall \lambda \in (0,1).$$

It follows that

$$\begin{aligned}(g \circ h)'(x; \bar{x} - x) &= \lim_{\lambda \downarrow 0} \frac{g[h(x + \lambda(\bar{x} - x))] - g(h(x))}{\lambda} \\ &\leq \lim_{\lambda \downarrow 0} \frac{g[h(x) + \lambda(h(\bar{x}) - h(x))] - g(h(x))}{\lambda} = g'[h(x); h(\bar{x}) - h(x)].\end{aligned}$$

By Theorem 23.3 of [12], $\partial g(h(x)) = \emptyset$ and $h(\bar{x}) \in \operatorname{ri} \operatorname{dom} g$ imply that the last expression equals $-\infty$, so that $\partial(g \circ h)(x) = \emptyset$ by Theorem 23.2 of [12]. □

Note that, according to Theorem 5, the condition $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$ is equivalent to the existence of an $x \in \operatorname{ri} \operatorname{dom} h$ such that $h(x) \in \operatorname{ri} \operatorname{dom} g$.

Corollary 3 *Let h be a K-convex function from X to U and C a convex set in U such that $K \subset \operatorname{rc} C$. Then for any $x \in h^{-1}(C)$*

$$N_{h^{-1}(C)}(x) \supset \mathrm{D}_K^* h(x) N_C(h(x)),$$

which holds as an equality if $\operatorname{ri} \operatorname{ran}_K h \cap \operatorname{ri} C \neq \emptyset$.

Proof. This follows by choosing $g = \delta_C$, and noting that $\delta_{h^{-1}(C)} = \delta_C \circ h$ and $N_S = \partial \delta_S$ for any convex set S. □

Using Theorems 7 and 11, we now obtain a chain rule for the K-Jacobian of vector-valued compositions:

Theorem 12 *Let h be a K_U-convex function from X to U and g a K_V-convex function from U to V such that $K_U \subset \operatorname{rc}_{K_V} g$. Then for any $x \in \operatorname{dom}(g \circ h)$*

$$\mathrm{D}_{K_V}^*(g \circ h)(x) \supset \mathrm{D}_{K_U}^* h(x) \mathrm{D}_{K_V}^* g(h(x)).$$

If $\operatorname{ri} \operatorname{ran}_{K_U} h \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$, then

$$\mathrm{D}_{K_V}^*(g \circ h)(x) = \mathrm{D}_{K_U}^* h(x) \mathrm{D}_{K_V}^* g(h(x))$$

and

$$\mathrm{D}_{K_V}(g \circ h)(x) = \operatorname{cl}(\mathrm{D}_{K_V} g(h(x)) \mathrm{D}_{K_U} h(x)).$$

Proof. By Theorem 4, $g \circ h$ is a K_V-convex function, so that by Theorem 7 we have for any $v^* \in K_V^*$

$$\mathrm{D}_{K_V}^*(g \circ h)(x) v^* = \partial \langle v^*, g \circ h \rangle (x).$$

Since $\langle v^*, g \circ h \rangle = \langle v^*, g \rangle \circ h$,

$$\mathrm{D}_{K_V}^*(g \circ h)(x) v^* = \partial(\langle v^*, g \rangle \circ h)(x).$$

Because $K_U \subset \mathrm{rc}_{K_V} g \subset \mathrm{rc}\,\langle v^*, g\rangle$ for any $v^* \in K_V^*$, we may apply the previous theorem to the functions $\langle v^*, g\rangle$ and h to obtain

$$\mathrm{D}^*_{K_V}(g \circ h)(x)v^* \supset \mathrm{D}^*_{K_U}h(x)\partial\langle v^*, g\rangle(h(x)) = \mathrm{D}^*_{K_U}h(x)\mathrm{D}^*_{K_V}g(h(x))v^*.$$

Since $\mathrm{dom}\,\langle v^*, g\rangle = \mathrm{dom}\, g$, this holds as an equality if $\mathrm{ri}\,\mathrm{ran}_{K_U} h \cap \mathrm{ri}\,\mathrm{dom}\, g \neq \emptyset$.

By Lemma 1 and Theorem 6.3 of [12], we obtain $\mathrm{ri}\,\mathrm{ran}\,\mathrm{D}_{K_U}h(x) = \mathrm{ri}\,T_{\mathrm{ran}_{K_U}h}(h(x))$ and $\mathrm{ri}\mathrm{dom}\,\mathrm{D}_{K_V}g((h(x)) = \mathrm{ri}T_{\mathrm{dom}\,g}(h(x))$, so that by Lemma 2, the condition $\mathrm{ri}\mathrm{ran}_{K_U}h \cap \mathrm{ri}\mathrm{dom}\, g \neq \emptyset$ implies that $\mathrm{ri}\,\mathrm{ran}\,\mathrm{D}_{K_U}h(x) \cap \mathrm{ri}\,\mathrm{dom}\,\mathrm{D}_{K_V}g(h(x)) \neq \emptyset$. Hence, by the equality for the adjoint Jacobians and by Theorem 2,

$$\begin{aligned}\mathrm{D}_{K_V}(g \circ h)(x) &= (\mathrm{D}^*_{K_U}h(x)\mathrm{D}^*_{K_V}g(h(x)))^* \\ &= (\mathrm{D}_{K_V}g(h(x))\mathrm{D}_{K_U}h(x))^{**} \\ &= \mathrm{cl}(\mathrm{D}_{K_V}g(h(x))\mathrm{D}_{K_U}h(x)).\end{aligned}$$

□

Corollary 4 *If A is a linear transformation from X to U, and g a K_V-convex function from U to V, then*

$$\mathrm{D}^*_{K_V}(gA)(x) \supset A^*\mathrm{D}^*_{K_V}g(Ax),$$

which holds as an equality if $\mathrm{ran}A \cap \mathrm{ri}\,\mathrm{dom}\, g \neq \emptyset$, *in which case we also have*

$$\mathrm{D}_{K_V}(gA)(x) = \mathrm{D}_{K_V}g(Ax)A.$$

Proof. Choosing $h = A$, $\mathrm{dom}\, h = X$ and $K_U = \{0\}$, we have $\mathrm{D}^*_{K_U}h(x) = A^*$ for all $x \in X$ and $\mathrm{dom}\,\mathrm{D}^*_{K_U}h(x) = U^*$, which by Theorem 2 implies the closedness of $\mathrm{D}_{K_V}g(Ax)\mathrm{D}_{K_U}h(x)$. □

Corollary 5 *If h is a K_U-convex function from X to U, and B is a linear transformation from U to V, then*

$$\mathrm{D}^*_{(BK_U)}(Bh)(x) = \mathrm{D}^*_{K_U}h(x)B^*,$$

and

$$\mathrm{D}_{(BK_U)}(Bh)(x) = \mathrm{cl}(B\mathrm{D}_{K_U}h(x)).$$

Proof. Choose $g = B$, $\mathrm{dom}\, g = U$ and $K_V = BK_U$. □

6 Application to convex programming

The classical approach to unconstrained minimization of a smooth function f is to find an expression for the gradient of f and set it equal to zero. This gives a necessary condition for a minimum point, in the form of an equation. Applicability of this method depends on the calculus rules used to find a manageable form for the gradient by expressing it in terms of more elementary functions defining f. Using the theory of generalized Jacobians of the previous sections, we are able to use a similar approach for convex functions which may be nonsmooth and defined only on a subset of the space of unknowns.

Consider convex programming problem of the form

$$\text{minimize} \quad f(x) = f_0(x) + g(h(x)),$$

where f_0 and g are extended real-valued convex functions on X and U respectively, and h is a K-convex function from X to U such that $K \subset \operatorname{rc} g$. We will refer to this problem as (P). We will say that (P) satisfies condition (C) if there exists an $x \in \operatorname{ri}\operatorname{dom} f_0 \cap \operatorname{ri}\operatorname{dom} h$ such that $h(x) \in \operatorname{ri}\operatorname{dom} g$.

Under (C), we have by Theorem 5, that

$$\operatorname{ri}\operatorname{dom}(g \circ h) = \{x \in \operatorname{ri}\operatorname{dom} h \mid h(x) \in \operatorname{ri}\operatorname{dom} g\},$$

so that $\operatorname{ri}\operatorname{dom} f_0 \cap \operatorname{ri}\operatorname{dom}(g \circ h) \neq \emptyset$ and $\operatorname{ri}\operatorname{ran}_K h \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$. Thus, using Theorem 23.8 of [12] and Theorem 11, we have

$$\partial f(x) \supset \partial f_0(x) + \partial(g \circ h)(x) \supset \partial f_0(x) + \mathrm{D}_K^* h(x) \partial g(h(x)),$$

with equalities if (C) is satisfied. Hence, the condition

$$0 \in \partial f_0(\bar{x}) + \mathrm{D}_K^* h(\bar{x}) \partial g(h(\bar{x})), \tag{1}$$

is always sufficient for an $\bar{x}$ to be a solution of (P), and if (C) is satisfied, it is also necessary. Condition (1) is equivalent to the existence of a $\bar{u}^*$ such that

$$0 \in \partial f_0(\bar{x}) + \mathrm{D}_K^* h(\bar{x}) \bar{u}^* \tag{2}$$

and

$$\bar{u}^* \in \partial g(h(\bar{x})). \tag{3}$$

Conditions (2)-(3) may also be derived from the Kuhn-Tucker conditions obtained via conjugate duality [13], and thus u^* may be interpreted as a Kuhn-Tucker vector for problem (P). To this end, we note that problem (P) corresponds to the generalized convex program associated with the function $F(u, x) = f_0(x) + g(h(x) - u)$. The corresponding Lagrangian is given by

$$\begin{aligned} L(u^*, x) &= \inf_u \{\langle u, u^* \rangle + F(u, x)\} \\ &= \begin{cases} +\infty & \text{if } x \notin \operatorname{dom} h \\ f_0(x) + \langle u^*, h(x) \rangle - g^*(u^*) & \text{if } x \in \operatorname{dom} h. \end{cases} \end{aligned}$$

Using Theorem 6.6 of [12] and Theorem 5, one can show that condition (C) implies that $0 \in \operatorname{ri}\operatorname{dom} \varphi$, where $\varphi(u) = \inf_x F(u, x)$. According to [12, Theorem 23.4] this implies that

φ is subdifferentiable at 0, which by Theorem 16 and Corollary 15A of [13] imply that the existence of an u^* such that

$$(0,0) \in \partial L(\bar{u}^*, \bar{x})$$

is a necessary and sufficient for optimality of $\bar{x}$. If g is closed and (C) holds, the above condition is equivalent to the conditions (2) and (3), by Theorems 23.5, 23.8 and Corollary 23.5.2 of [12]. Using reasoning similar to that in the end of the proof of Theorem 11, the closedness assumption on g could be relaxed.

Note that in the first approach to deriving (2)-(3), based only on differentials, the closedness of g is irrelevant, since the subdifferentiation rules [12, Theorem 23.8] and Theorem 11 do not depend on the closedness of the functions involved.

For $h(x) = Ax$ and $K = \{0\}$, condition (C) reduces to the condition $R(A) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$, and we obtain the Fenchel-Rockafellar model of [12, Section 31] where $f(x) = f_0(x) + g(Ax)$. On the other hand, when $g = \delta_C$ for some convex set $C \subset U$ such that $K \subset \operatorname{rc} C$, problem (P) may be written with explicit constraints:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & h(x) \in C, \end{array}$$

and the condition (3) reduces to $\bar{u}^* \in N_C(h(\bar{x}))$. If C is the cone K, this reduces further to $\bar{u}^* \in \partial N_K(h(\bar{x}))$, which is equivalent to condition

$$h(\bar{x}) \in K, \quad \bar{u}^* \in K^*, \quad \langle \bar{u}^*, h(\bar{x}) \rangle = 0. \tag{4}$$

When $U = \Re^m$ and K is of the form $K = K_1 \times \cdots \times K_m$, where $K_i = \Re_-$ for $i \in I \subset \{1, \ldots, m\}$, and $K_i = \{0\}$ for $i \in E = \{1, \ldots, m\} \setminus I$, the K-convexity of h means that $h = (h_1, \ldots, h_m)$, where h_i is a real-valued convex function for all $i \in I$, and real-valued affine function for all $i \in E$. For this model, condition (C) reduces to the existence of

$$\begin{array}{rl} & x \in \operatorname{ri}\operatorname{dom} f_0 \cap \bigcap_{i \in I} \operatorname{ri}\operatorname{dom} h_i \\ \text{such that} & h_i(x) < 0 \ \forall i \in I, \text{ and } h_i(x) = 0 \ \forall i \in E. \end{array} \tag{5}$$

The condition (4) takes the familiar form

$$\begin{array}{ll} h_i(\bar{x}) \leq 0, \quad \bar{u}_i^* \geq 0, \quad \bar{u}_i^* h_i(\bar{x}) = 0 & \forall i \in I, \\ h_i(\bar{x}) = 0 & \forall i \in E, \end{array}$$

and the second term in (2) may then be written as

$$\begin{aligned} \mathrm{D}_K^* h(\bar{x}) \bar{u}^* &= \sum_{i \in I} \mathrm{D}_{\Re_-}^* h_i(\bar{x}) \bar{u}_i^* + \sum_{i \in E} \mathrm{D}_{\{0\}}^* h_i(\bar{x}) \bar{u}_i^* \\ &= \sum_{i \in I \cup E} \partial(\bar{u}_i^* h_i)(\bar{x}). \end{aligned}$$

For i such that $\bar{u}_i^* = 0$, $\bar{u}_i^* h_i$ should be interpreted as the indicator function of $\operatorname{dom} h_i$, so that $\partial(\bar{u}_i^* h_i)(\bar{x}) = N_{\operatorname{dom} h_i}(\bar{x})$. For i with $\bar{u}_i^* \neq 0$, we simply have $\partial(\bar{u}_i^* h_i)(\bar{x}) = \bar{u}_i^* \partial h_i(\bar{x})$.

In summary, the existence of a $\bar{u}^*$ such that

(a) $h_i(\bar{x}) \le 0, \;\; \bar{u}_i^* \ge 0, \;\; \bar{u}_i^* h_i(\bar{x}) = 0 \;\; \forall i \in I,$

(b) $h_i(\bar{x}) = 0 \; \forall i \in E,$

(c) $0 \in \partial f_0(\bar{x}) + \sum_{i \in I \cup E} \partial(\bar{u}_i^* h_i)(x),$

is always sufficient for an $\bar{x}$ to be optimal solution for the convex programming problem

$$\begin{array}{lllll} \text{minimize} & f_0(x) & & & \\ \text{subject to} & h_i(x) & \le & 0 & \forall i \in I \\ & h_i(x) & = & 0 & \forall i \in E. \end{array}$$

If the "constraint qualification" (5) is satisfied, these conditions are also necessary. This strengthened version of the standard Slater condition is a consequence of allowing constraint functions with arbitrary (convex) domains. It enables us to express the subdifferential of the Lagrangian with respect to x in terms of the subdifferentials of the objective and constraint functions.

As a last example, we derive a formula for the subdifferential of the convex function f given by

$$f(x) = \max_i h_i(x),$$

where h is as in the previous example. Defining $g(u) = \max_i u_i$, we have $f = g \circ h$, $\operatorname{dom} g = \Re^m$ and $\operatorname{rc} g = \Re^m_- \supset K$. By Theorems 11 and 8, we have

$$\begin{aligned} \partial f(x) &= \mathrm{D}_K^* h(x) \partial g(h(x)) \\ &= \bigcup \left\{ \sum_{i \in I \cup E} \partial(u_i^* h_i)(x) \;\middle|\; u^* \in \partial g(h(x)) \right\} \\ &= \bigcup \left\{ \sum_{i \in I \cup E} \partial(u_i^* h_i)(x) + \sum_{i \in I \cup E} N_{\operatorname{dom} h_i}(x) \;\middle|\; u^* \in \partial g(h(x)) \right\}. \end{aligned}$$

Noting that $\partial g(u) = \operatorname{co}\{e_i \mid u_i = g(u)\}$, where e_i denotes the ith column of the identity matrix, we obtain the formula

$$\partial f(x) = \operatorname{co} \bigcup \{\partial h_i(x) \mid h_i(x) = g(h(x))\} + \sum_{i \in I \cup E} N_{\operatorname{dom} h_i}(x).$$

Using the calculus rules of the previous section, one can derive subdifferential formulas for a wide class of convex functions. The chain rule given in Theorem 12 and its special case in Theorem 11 considerably extend the applicability of the subdifferential calculus of convex analysis. As demonstrated above, the calculus rules are now general enough to provide a straightforward and unified way to derive optimality conditions for standard models in convex programming. The more structure a problem has, the more detailed conditions one may be able to obtain.

Most of the results of this paper may be sharpened in the case of polyhedral convexity: one would simply invoke polyhedral refinements of the cited results of [12]. One could also attempt to generalize the results here to cases where the spaces X and U may have infinite dimension. Another research direction might be extending the theory to nonconvex functions. In this case, there are various possibilities for defining tangent and normal cones; see for example [2, 15]. To obtain good local approximations to nonconvex sets, one may have to use nonconvex normal and tangent cones, thus sacrificing the perfect duality between them.

We have not paid any attention to the continuity properties of vector-valued functions. However, something corresponding to lower-semicontinuity of real-valued functions will be important when one wishes to prove the closedness of a real-valued function obtained by composition from vector-valued functions. This and other related issues will be addressed in subsequent papers.

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