Alex Reznik
Problem 3.41

a. We note that such an $\mathbf{R}$ matrix can be written in the form

$$\mathbf{R} = (1-\rho)\mathbf{I} + \rho\mathbf{1}$$

where $\mathbf{1}$ is a matrix with all of its entries equal to 1. Then for any K-vector $\mathbf{x}$

$$\mathbf{x^T R x} = (1-\rho)\sum_{i=1}^{K} x_i^2 + \rho\left(\sum_{i=1}^{K} x_i\right)^2$$

Thus, $\mathbf{x^T R x}$ is a weighted sum of squares and is therefore guaranteed to be positive if the weights on all the squares are positive. Thus, $\mathbf{R}$ is positive definite for $0 < \rho < 1$. $\mathbf{R}$ is not positive definite for $\rho = 1$ (and for $\rho > 1$, although this is not possible in the context of CDMA). This can be seen by observing that $\sum_{i=1}^{K} x_i$ can always be made 0 by an appropriate choice of the vector x, making $\mathbf{x^T R x}$ negative or 0 because the weight on the other squares becomes negative or 0.

To extend this result to non-positive ρ, we note that Gershgorin's disc theorem guarantees that the eigenvalues of $\mathbf{R}$ lie in the interval $[1-\rho(K-1), 1+\rho(K-1)]$. Thus, as long as $|\rho| < 1/(K-1)$, all the eigenvalues are guaranteed to be positive by this theorem and therefore $\mathbf{R}$ is guaranteed to be positive definite.

Lastly, consider $\mathbf{x} = [1, 1, \ldots, 1]^T$. Then $\mathbf{x^T R x} = K(1 + (K-1)\rho)$, which is less then or equal to 0 for $\rho \leq -1/(K-1)$ and therefore $\mathbf{R}$ is not positive definite for such ρ.

Putting it all together, we get that $\mathbf{R}$ is positive definite for

$$-\frac{1}{K-1} < \rho < 1$$

b. The desired expression is simply a special case of the general expression derived in (3.90). Consider the summation

$$\sum_{j\neq k} e_j \frac{A_j}{\sigma}\rho_{jk}$$

which appears inside the Q-function in (3.90). Since $A_j = A\ \forall j$ and $\rho_{jk} = \rho\ \forall j, k : j \neq k$, this summation takes on values in the set $\big\{-(K-1), -(K-1)+2, \ldots, (K-1)-2, (K-1)\big\}$. There are $\binom{K-1}{n}$ distinct combinations of bits $j \neq k$ that makes the summation take on the value $(K-1) - 2n$.

Thus, the argument inside the Q-function in (3.90) collapses to

$$\frac{A}{\sigma} + \frac{A}{\sigma}\rho(K-1-2n) \quad n = 0, \ldots, K-1$$

For each n, there are $\binom{K-1}{n}$ such Q-functions in the complete summation and since each one is independent of the values of $e_1, \ldots, e_K$, the $K-1$ summations of (3.90) collapse to a single summation over n, reducing (3.90) to the desired result.

c. Substituting for $A_j = A \; \forall j$ and $\rho_{jk} = \rho \; \forall j, k : j \neq k$ into (3.93) we get:

$$\tilde{P}_k^c(\sigma) = Q\left(\frac{A}{\sqrt{\sigma^2 + (K-1)\rho^2 A^2}}\right) = Q\left(\frac{1}{\sqrt{\sigma^2/A^2 + (K-1)\rho^2}}\right)$$

d. First,

$$\lim_{\sigma \to 0} \tilde{P}_k^c(\sigma) = \lim_{\sigma \to 0} Q\left(\frac{1}{\sqrt{\sigma^2/A^2 + (K-1)\rho^2}}\right) = Q\left(\frac{1}{\sqrt{(K-1)\rho^2}}\right)$$

Next we note that we can pick ρ large enough so that the probability of error is exactly one-half. In fact a value of $\rho = 1$ will do for all K. To see this, we first note that

$$\lim_{\sigma \to 0} Q\left(\frac{x}{\sigma}\right) = \begin{cases} 0 & \text{for } x > 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x < 0 \end{cases}$$

Then, for $\rho = 1$ and as $\sigma \to \infty$,

$$Q\left(\frac{A}{\sigma}(1 + \rho(K-1-2n))\right) = \begin{cases} 0 & \text{when } n < \frac{K}{2} \\ \frac{1}{2} & \text{when } n = \frac{K}{2} \text{ (occurs IFF } K \text{ is even)} \\ 1 & \text{when } n > \frac{K}{2} \end{cases}$$

and the expression for probability of error given in part b then becomes

$$\lim_{\sigma \to 0} P_k^c(\sigma) = \frac{1}{2^{K-1}} \frac{1}{2} \sum_{n=0}^{K-1} \binom{K-1}{n} = \frac{1}{2^{K-1}} \frac{1}{2} 2^{K-1} = \frac{1}{2}$$

However, because $\dfrac{1}{\sqrt{(K-1)\rho^2}}$ is strictly positive for $\rho = 1$, $\lim_{\sigma \to 0} \tilde{P}_k^c(\sigma) < 1/2$, which proves the desired result.

e. We recall from part d. that for $\rho = 1$ $\lim_{\sigma \to 0} P_k^c(\sigma) = 1/2$. Clearly, we also have $\lim_{\sigma \to \infty} P_k^c(\sigma) = 1/2$. It is also fairly easy to convince oneself that there are values of σ for which $P_k^c(\sigma) < 1/2$. Therefore, for $\rho = 1$ $P_k^c(\sigma)$ must be non-monotic for all K.

The explanation for this phenomenon is the same as the explantion given for the two user case (Section 3.4.1). The multiuser interference is so high, that, in the absence of noise, the sign of the output of any user's mathched filter is completely determined by the sign of the summation of all the signals sent by all the users - which is positive or negative with equal probability. If the result of the summation is exactly 0, an error is made with probability 1/2.

When some noise is present, those values that wind up close to 0, but in error from the point of view of user k, may sometimes get pushed across the boundary by the additive noise - thus eliminating the error. This explains how a moderate amount of additive noise can reduce the error probability in this anomalous scenario.