Notes on the Ahlfors Mapping of a Multiply Connected Domain

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These notes follow the exposition of the Ahlfors mapping via the Szegö and Garabedian kernels as presented in S. Bell's book *The Cauchy Transform, Potential Theory, and Conformal Mapping*. I've rearranged things to suit my own preferences, but it's basically taken verbatim from Bell's book.

Extremal problems and the Riemann mapping theorem

To explain Ahlfors's result let's begin with

> **Riemann Mapping Theorem** If Ω is a simply connected domain with more than one boundary point then there is a conformal mapping f of Ω onto the unit disk $\mathbf{D}$. The mapping is uniquely determined by specifying a point $a \in \Omega$ with $f(a) = 0$ and $f'(a) > 0$.

The uniqueness follows from Schwarz's lemma. The standard proof of existence, as given by Koebe in 1907, is through the solution of an extremal problem:

Consider the class of functions $\mathcal{F}$ analytic and injective in Ω with

(a) $|g| < 1$ on Ω
(b) $g(a) = 0$, $g'(a) > 0$

Then the function solving the extremal problem

$$\max_{g \in \mathcal{F}} g'(a)$$

maps Ω conformally onto $\mathbf{D}$.

Now, answer quickly: What happens if we drop the condition that the competing functions are injective? That is, what happens if we maximize over the larger class of functions that are simply analytic and bounded by 1 in Ω? The answer is the same – that the solution to this new (relaxed?) extremal problem is again a conformal mapping of Ω onto $\mathbf{D}$ (unique if normalized as above). This is *not obvious* – certainly Koebe's proof cannot be easily modified to eliminate the assumption of injectivity. That an apparently less restrictive extremal problem has the same solution as the original problem emerges as a special case of Ahlfors's theorem.[1]

*With apologies to Steven Bell

[1] I don't know other (easy) ways of seeing this, except that I think it can also be deduced through the connection between the Riemann mapping theorem and the Bergman kernel, and the extremal properties of the Bergman kernel. That may avoid some of the issues of smoothness at the boundary that come up in Bell's approach. I suppose it also follows from 'the principle of the hyperbolic metric', which is through potential theory. It's all related.

'The true equivalent of Schwarz's lemma for single-valued functions in a multiply-connected domain.'

The quote is from Ahlfors's 1946 paper *Bounded Analytic Functions*. He considers precisely the extremal problem above in the setting of multiply-connected domains. Here's a partial statement of his result (we'll give a more complete version later).

> **Ahlfors Mapping Theorem, first version** Let Ω be a domain of connectivity $n \geq 1$, none of whose boundary components reduce to a point. Fix a point $a \in \Omega$ and consider the extremal problem
> $$\max |f'(a)|$$
> where f is analytic (and single-valued) in Ω with $|f| < 1$. The solution of this problem is an $n : 1$ map onto $\mathbf{D}$.

We see that the Riemann mapping theorem corresponds to $n = 1$, and, for that case, that Koebe's extremal problem can be relaxed to the family of functions that are analytic but need not be one-to-one. There's a slight catch here. Ahlfors needs a certain amount of boundary smoothness to form integrals over $\partial\Omega$, and he uses some *preliminary* conformal mapping to assume that $\partial\Omega$ consists of analytic Jordan curves (a standard technique when working with finitely connected domains). This *requires* the Riemann mapping theorem! (The general methods in Bell's book assume C^∞ boundaries.) To what extent these assumptions can be weakened or worked around I don't know. (And I don't really care – it can't be an issue for our ultimate concerns, considering that we're going to take a discrete approximation to the boundary anyway!)

The problem of maximizing the derivative at a point is precisely the character of Schwarz's lemma, whence the opening quote. It's interesting that the same type of extremal problem was considered in generality for various classes of analytic functions in the famous paper by Ahlfors and Beurling on *Conformal Invariants and Function Theoretic Null Sets*. Indeed the 'null set' aspect was already important for Ahlfors in the paper where the theorem, above, was proved (Painlevé's problem).[2]

We note one very simple property of the extremal function: The extremal must vanish at the point a. For if f takes values in $\mathbf{D}$ then so does

$$f_1(z) = \frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)},$$

for which

$$f_1'(z) = \frac{f'(z)}{1 - \overline{f(a)}f(z)}.$$

Thus if $f(a) \neq 0$ then

$$|f_1'(a)| = \frac{|f'(a)|}{1 - |f(a)|^2} > |f'(a)|$$

We also note that it's possible to modify the extremal problem in ways that, depending on the approach, can make it easier to work with. Obviously all rotations $e^{i\theta}f(z)$ of competing functions

[2] And what about maximizing other functionals? The Schwarzian at a point? The pre-Schwarzian at a point?

are again competing functions, so we can first assume the problem is to maximize $f'(a)$ among functions with $f'(a) > 0$. Also, if $f(z)$ has values in $\mathbf{D}$ then $g(z) = \frac{4}{\pi}\tan^{-1} f(z)$ has values in $-1 < \operatorname{Re} z < 1$. If $f(a) = 0$ and $f'(a) > 0$ then

$$g'(a) = \frac{4}{\pi} f'(a) > 0.$$

Hence the original extremal problem is equivalent to maximizing $\operatorname{Re} g'(a)$ over the class of analytic functions in Ω satisfying $|\operatorname{Re} g(z)| < 1$.

Moving on, or, 'Everyone wants a *method*'

Ahlfors didn't stop with the plane domain case. He wrote a follow-up paper on open Riemann surfaces ('One of my major papers', he said in a commentary on the paper in his collected works.) where he moved toward a more general formulation and treatment of extremal problems. He emphasized the use of differentials, which come in naturally (and necessarily) for Riemann surfaces. Indeed, the solutions of the extremal problems he considered are 'Schottky differentials', both for Riemann surfaces and for plane domains.

Meanwhile, back in the plane, Garabedian, who was Ahlfors's student at the time, advanced the use of kernel functions as an alternate approach to Ahlfors's original arguments. This pushed things more in the direction of functional analysis, but still very much in a 'complex variable' setting. In fact, the classical masters (Schiffer, Garabedian) referred to the circle of ideas for this and other problems as 'the method of contour integration', where orthogonality is put to use through the vanishing of certain integrals over the boundary of a domain. In the right hands this results in interesting identities that are often coupled with residue calculations.

Incidentally, Ahlfors credits Garabedian with the observation that 'the relevant extremal problems occur in pairs connected by a sort of duality' but, as far as I can tell, it's not made explicit what is the problem and what is the dual. Garabedian's approach to the Ahlfors mapping was through the Szegö kernel and the (later to be named) Garabedian kernel. This is also covered in Bergman's book on *The Kernel Function and Conformal Mapping* and in the last section of the last chapter of Nehari's book *Conformal Mapping*.

A more definitive change in method, certainly in point of view, occurred with the work of Stein and of Kerzman and Stein in the '70's. There the central problems are 'boundary values of holomorphic functions', in one or several complex variables, and the techniques draw from real variables and functional analysis. Part of the impetus was also the real-variable extensions of Hardy spaces to $\mathbf{R}^n$ by Fefferman, Stein and others.

At the very least the Kerzman-Stein arguments 'look different' even though one can recover the classical results, as we will. It seems that the formulations in terms of extremal problems are not so front and center, though perhaps a version is still there in the use of orthogonal projections (where somewhere hidden in the slick set-up is the minimum distance from a function to a subspace of an appropriate Hilbert space). The context is also seemingly more general; this may be somewhat a matter of taste and may take awhile to play out – but I like it.

'Functional analysis' means ...

The new method is self-consciously one of functional analysis. That means setting up the right spaces and the right operators on those spaces. We won't do all of this all at once – that's what these notes (and Bell's book) are about – but here are a few fundamental notions. Before anything else, we will, with Bell, adopt the following

Standing assumption Ω is a bounded, finitely connected domain with C^∞ Jordan boundary curves.

At various times we'll need to say a few things about how the boundary is traced out, introducing a unit tangent vector field.

Some spaces We let $A^\infty(\Omega)$ be the analytic functions in Ω that are C^∞ on $\overline{\Omega}$. Thus $u \in A^\infty(\Omega)$ if u is analytic in the interior and all its derivatives extend continuously to $\partial\Omega$. If $u \in A^\infty(\Omega)$ then according to the Cauchy integral formula

$$u(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega$$

(Assuming that u is C^∞ on the boundary is overkill.)

You may have thought there was nothing new to understand about the Cauchy integral formula, but you'd be surprised. The first step is to set up the appropriate spaces on $\partial\Omega$. One is $C^\infty(\partial\Omega)$ and the second is

$A^\infty(\partial\Omega)$, the set of boundary values for functions in $A^\infty(\Omega)$. We can obviously identify $A^\infty(\partial\Omega)$ with $A^\infty(\Omega)$, but it's sometimes convenient to keep the two spaces separate.

From the functional analyst's perspective these aren't particularly useful spaces (they're not complete). Thus we define:

$H^2(\partial\Omega)$ is the closure in $L^2(\partial\Omega)$ of $A^\infty(\partial\Omega)$

This is an alternate definition for the classical Hardy space. Bell shows that this really is the same as the usual Hardy space of analytic functions on Ω. That takes some work.

As a closed subspace of $L^2(\partial\Omega)$, $H^2(\partial\Omega)$ is a Hilbert space with the usual inner product;

$$(u, v) = \int_{\partial\Omega} u(\zeta)\overline{v(\zeta)}\, |d\zeta|. \quad \text{(Note the integration is with respect to arclength.)}$$

Some operators We've already seen one operator that will be central to our concerns – the one given by the Cauchy integral – but rather than start with an analytic function in Ω we shift attention to the boundary. Let $u \in C^\infty(\partial\Omega)$. The *Cauchy transform* of u is

$$(\mathcal{C}\, u)(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega.$$

Clearly this defines an analytic function on Ω.

What about the boundary values of $\mathcal{C}\, u$? One of our first results will be an important foundational fact (if not so exciting itself), namely that $\mathcal{C}\, u$ extends to be C^∞ on $\partial\Omega$, or put another way, that the Cauchy transform maps $C^\infty(\partial\Omega)$ into $A^\infty(\Omega)$. However, for C^∞ functions $\mathcal{C}\, u$ is *not* equal to u on the boundary. There are different ways of expressing what does happen. One is a classical result of Plemelj that reads

$$(\mathcal{C}\, u)(z) = \frac{1}{2}u(z) + \mathcal{H}u(z), \quad z \in \partial\Omega,$$

where $\mathcal{H}u$ is the Hilbert transform on $\partial\Omega$:

$$(\mathcal{H}u)(z) = \text{p.v.}\frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta. \quad \text{(principal value)}$$

Later we'll also look at another way of representing the boundary values of $\mathcal{C}\,u$ due to Bell.

At the heart of the Kerzman-Stein development (and Bell's reworking of it) is a method for extending $\mathcal{C}$ to be an operator on $L^2(\partial\Omega)$, including:

- A study of the L^2-adjoint $\mathcal{C}^*$ of $\mathcal{C}$, showing that
- $\mathcal{C} - \mathcal{C}^* = \mathcal{A}$ is a better operator than either $\mathcal{C}$ or $\mathcal{C}^*$ in the sense that

$$(\mathcal{A}u)(z) = \int_{\partial\Omega} A(z,\zeta)u(\zeta)\,d\zeta, \quad z \in \partial\Omega,\ u \in L^2(\partial\Omega),$$

  where $A(z,\zeta)$ is C^∞ as a function of (z,ζ) on $\partial\Omega \times \partial\Omega$.

Bell refers to this result as showing that $\mathcal{C}$ is 'almost' self-adjoint.

If $u \in L^2(\partial\Omega)$ then one can show that

$$H(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\,d\zeta$$

is an analytic function on Ω with L^2-boundary values $\mathcal{C}\,u$. Thus we write

$$(\mathcal{C}\,u)(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\,d\zeta$$

for $H(z)$. In fact, there is a one-to-one correspondence between functions in $H^2(\partial\Omega)$ and analytic functions on Ω arising as their Cauchy integrals

The Szegö and Garabedian projections Since $H^2(\partial\Omega)$ is (by definition) a closed subspace of $L^2(\partial\Omega)$ we can consider the orthogonal decomposition

$$L^2(\partial\Omega) = H^2(\partial\Omega) \oplus H^2(\partial\Omega)^\perp$$

and the corresponding projections. The projection $P\colon L^2(\partial\Omega) \to H^2(\partial\Omega)$ is (now) called the *Szegö projection*. We'll see that P maps $C^\infty(\partial\Omega)$ into itself. The projection $P^\perp\colon L^2(\partial\Omega) \to H^2(\partial\Omega)^\perp$ is called the *Garabedian projection*.

This particular orthogonal decomposition of $L^2(\partial\Omega)$ furnishes a really useful way of splitting L^2-functions into components. The result is essentially due to Schiffer. Let T be the unit tangent vector field to $\partial\Omega$ pointing in the direction of the usual orientation.

> **Theorem [Orthogonal decomposition of $L^2(\partial\Omega)$]** A function $u \in L^2(\partial\Omega)$ can be written as
> $$u = h + \overline{TH},$$
> where $h = Pu \in H^2(\partial\Omega)$, $\overline{TH} \in H^2(\partial\Omega)^\perp$, and $H = P(\bar{u}\overline{T})$.

We'll give a complete proof of this after developing further properties of the Cauchy transform, but let's see now why h and $\overline{TH}$ are orthogonal. The tangent vector T comes in because the integration in the inner product is with respect to arclength and

$$d\zeta = T|d\zeta|, \quad \text{because} \quad \frac{d\zeta}{dt} = \frac{d\zeta}{ds}\frac{ds}{dt}.$$

Note also that since $|T| = 1$ we have

$$\bar{T} d\zeta = |d\zeta|.$$

With this, the orthogonality is easy: Let h and H be in $H^2(\partial\Omega)$ and let h_i and H_i be sequences in $A^\infty(\Omega)$ converging in $L^2(\partial\Omega)$ to h and H, respectively. Then

$$(h, \overline{TH}) = \lim_{i\to\infty} (h_i, \overline{TH_i}) = \lim_{i\to\infty} \int_{\partial\Omega} h_i H_i T \, |d\zeta| = \lim_{i\to\infty} \int_{\partial\Omega} h_i H_i \, d\zeta,$$

and the last integral is zero because of Cauchy's theorem!

The Szegö kernel We can realize the Szegö projection P as an integral operator with a kernel and relate it to the Cauchy transform. For $a \in \Omega$ and $z \in \partial\Omega$ let $C_a(z)$, the *Cauchy kernel*, be defined by

$$\overline{C_a(z)} = \frac{1}{2\pi i} \frac{T(z)}{z - a}.$$

We take complex conjugates because we want to use C_a as the second factor in an inner product. That is, C_a is the kernel that defines the (L^2) Cauchy integral via:

$$(\mathcal{C}\, u)(a) = (u, C_a), \quad u \in L^2(\partial\Omega).$$

If $h \in H^2(\partial\Omega)$ then using the orthogonal decomposition of $L^2(\partial\Omega)$ we can write

$$(h, C_a) = (h, PC_a).$$

Now let

$$S_a = PC_a \in H^2(\partial\Omega).$$

S_a is called the *Szegö kernel*. It has a remarkable 'reproducing property': Identify h with the analytic function in Ω defined by its Cauchy integral. Then we may write

$$h(a) = (h, C_a) = (h, PC_a) = (h, S_a).$$

We'll prove later that P maps $C^\infty(\partial\Omega)$ into itself. If again we identify S_a with the analytic function $\mathcal{C}\, S_a$ in Ω, then we can say that $S_a(z) = (PC_a)(z)$ is in $A^\infty(\Omega)$ as a function of $z \in \Omega$ when $a \in \Omega$ is fixed. It's customary to write

$$S_a(z) = S(z, a),$$

and one can show (we won't: Bell, p. 22) that $S(z, a)$ is continuous on $\Omega \times \Omega$ and antiholomorphic in a.

The Szegö kernel is also Hermitian, in the sense

$$S(z, a) = \overline{S(a, z)}.$$

To see this, we have

$$S(z, a) = S_a(z) = (S_a, S_z) = \overline{(S_z, S_a)} = \overline{S(a, z)}.$$

Using this we can write the reproducing property as

$$h(a) = \int_{\partial\Omega} S(a, z) h(z) \, |dz|, \quad h \in H^2(\partial\Omega);$$

note that the integration is in terms of arclength on $\partial\Omega$. This also expresses the Szegö projection as an integral:

$$(Pu)(a) = (Pu, S_a) = (u, PS_a) = (u, S_a) = \int_{\partial\Omega} S(a,z)h(z)\,|dz|.$$

Finally, note that

$$S(a,a) = (S_a, S_a) = ||S_a||^2 = \int_{\partial\Omega} |S_a|^2\,|dz|.$$

In particular, $S(a,a) > 0$. (It can't be identically zero.)

Here's a quick summary:

For $a \in \Omega$ the Szegö kernel S_a is the projection of the Cauchy kernel C_a onto $H^2(\partial\Omega)$. Thus $S_a \in H^2(\partial\Omega)$, and actually $S_a(z)$ is in $A^\infty(\Omega)$. We write $S_a(z) = S(z,a)$ and we have the reproducing property:

$$h(a) = \int_{\partial\Omega} S(a,z)h(z)\,|dz|, \quad h \in H^2(\partial\Omega);$$

The Garabedian kernel The Szegö kernel is the component in $H^2(\partial\Omega)$ when the orthogonal decomposition of $L^2(\partial\Omega)$ is used for the Cauchy kernel. Let's see what happens with the component in $H^2(\partial\Omega)^\perp$. According to the orthogonal decomposition theorem we can write

$$C_a = PC_a + \overline{TH_a} = S_a + \overline{TH_a}.$$

Here H_a is in $A^\infty(\Omega)$. Take complex conjugates and use the Hermitian symmetry of S_a:

$$\frac{1}{2\pi i}\frac{T(\zeta)}{\zeta - a} = S(a,\zeta) + H_a(\zeta)T(\zeta).$$

The *Garabedian kernel* is defined to be

$$L_a(\zeta) = L(\zeta,a) = \frac{1}{2\pi}\frac{1}{\zeta - a} - iH_a(\zeta).$$

Note that, like S, L is smooth on the boundary.

We plug this expression back into the formula for the orthogonal decomposition of the Cauchy kernel to get an equation relating the Szegö and Garabedian kernels:

Fundamental identity for S and L

$$S(a,\zeta) = \frac{1}{i}L(\zeta,a)T(\zeta), \quad a \in \Omega,\ \zeta \in \partial\Omega.$$

Observe then that $|S| = |L|$ (on the boundary).

Using this identity and the reproducing property of the Szegö kernel we can essentially write the Garabedian projection $P^\perp$ in terms of the Garabedian kernel. Since $P^\perp u = \overline{HT}$ for $u \in L^2(\partial\Omega)$ and $H = P(\overline{u}\overline{T})$, we find that

$$\begin{aligned} H(a) &= P(\overline{u}\overline{T})(a) \\ &= \int_{\partial\Omega} S(a,\zeta)\overline{u(\zeta)T(\zeta)}\,|d\zeta| \\ &= \frac{1}{i}\int_{\partial\Omega} L(\zeta,a)\,|d\zeta|. \end{aligned}$$

The identity has another consequence in light of the orthogonal decomposition of $L^2(\partial\Omega)$. Multiplying both sides of the equation by $\overline{T}$ we have

$$L(\zeta, a) = i\overline{S(a,\zeta)T(\zeta)}.$$

Hence $L(\zeta, a) \in H^2(\partial\Omega)^\perp$ and is thus equal to its projection onto $H^2(\partial\Omega)^\perp$. Now write

$$L(\zeta, a) = \frac{1}{2\pi}\frac{1}{\zeta - a} - iH_a(\zeta) = G_a(\zeta) - iH_a(\zeta).$$

Since $H_a \in A^\infty(\Omega)$ its projection on $H^2(\partial\Omega)^\perp$ is 0, and so

$$L(\zeta, a) = P^\perp(L(\cdot, a))(\zeta) = P^\perp(G_a - iH_a)(\zeta) = P^\perp(G_a)(\zeta).$$

In words:

> The Garabedian kernel is the projection onto $H^2(\partial\Omega)^\perp$ of the 'basic simple pole', $1/(2\pi(\zeta - a))$. In particular, S_a and L_a are orthogonal in $L^2(\partial\Omega)$.

This *isn't* projecting the Cauchy kernel as we did to get the Szegó kernel. The Cauchy kernel has the tangent vector T in its definition, and that makes a big difference.

The Szegö and Garabedian kernels for the disk We can find $S(z, a)$ and $L(z, a)$ directly for the unit disk $\mathbf{D}$ via the equation relating the two kernels and the orthogonal decomposition of the Cauchy kernel. The unit tangent vector field to the unit circle is just $T(\zeta) = i\zeta$. Hence

$$C_a(z) = -\frac{1}{2\pi i}\frac{\overline{T(z)}}{\bar{z} - \bar{a}} = \frac{1}{2\pi}\frac{\bar{z}}{\bar{z} - \bar{a}}.$$

Now $\bar{z} = 1/z$ on $|z| = 1$, so this can be put in the form

$$C_a(z) = \frac{1}{2\pi}\frac{1}{1 - \bar{a}z}.$$

This expression makes it obvious that $C_a(z)$ is an analytic function of z, and therefore that $PC_a = C_a$. Hence

$$S_a(z) = (PC_a)(z) = \frac{1}{2\pi}\frac{1}{1 - \bar{a}z}.$$

Furthermore, since $S_a = PC_a$ the H_a term in the orthogonal decomposition of C_a must be zero. Thus from

$$L(z, a) = \frac{1}{2\pi}\frac{1}{z - a} - iH_a(z)$$

we conclude that

$$L(z, a) = \frac{1}{2\pi}\frac{1}{z - a}.$$

Riemann redux:

Look carefully at the formulas for the Szegö and Garabedian kernels for the disk, and look at them together. Their ratio is

$$\frac{S(z, a)}{L(z, a)} = \frac{z - a}{1 - \bar{a}z},$$

a Möbius transformation of the $\mathbf{D}$ onto itself. And now look:

Theorem [The Riemann mapping via S and L] Let Ω be a simply connected domain with C^∞ boundary and let $a \in \Omega$. Then the Riemann mapping function f of Ω onto $\mathbf{D}$ with $f(a) = 0$ and $f'(a) > 0$ is given by

$$f(z) = \frac{S(z,a)}{L(z,a)},$$

where $S(z,a)$ and $L(z,a)$ are the Szegö and Garabedian kernels of Ω.

The proof depends on comparing the boundary behavior of f to that of S and L; in particular it depends on knowing that f extends continuously to the boundary. This is Caratheodory's theorem, which we'll assume.[3]

Let

$$\nu(z) = \frac{S(z,a)}{f(z)} \quad \text{and} \quad \mu(z) = L(z,a)f(z), \quad z \in \Omega.$$

Since $S(a,a) > 0$, in particular $S(a,a)$ is non-zero, the function $\nu(z)$ is meromorphic in Ω with a single simple pole at a. The function $\mu(z)$ is analytic in Ω because $L(z,a)$ has a simple pole at a. Both ν and μ extend continuously to the boundary. (We know this for S and L from the theory we've developed, and here is where we use Caratheodory's theorem for f.)

Now $f = 1/\bar{f}$ on $\partial\Omega$. Using the fundamental identity relating the Szegö and Garabedian kernels, we then have for $z \in \partial\Omega$,

$$\begin{aligned}
\overline{\nu(z)} &= \frac{\overline{S(z,a)}}{\overline{f(z)}} \\
&= S(a,z)f(z) \\
&= \frac{1}{i}L(z,a)T(z)f(z) \\
&= \frac{1}{i}\mu(z)T(z),
\end{aligned}$$

This is the same identity satisfied by S and L, and it has similar consequences. That is, taking the conjugate,

$$\nu(z) = i\overline{\mu(z)}\,\overline{T(z)}.$$

and this says that ν is orthogonal to $H^2(\partial\Omega)$. If we write, as we did for L,

$$\nu(z) = \frac{1}{2\pi}\frac{c}{z-a} + \text{analytic} = cG_a(z) + \text{analytic},$$

then

$$\nu(z) = P^\perp(cG_a)(z) = cL(z,a).$$

Thus we have

$$cL(z,a) = \nu(z) = \frac{S(z,a)}{f(z)}, \quad \text{or} \quad f(z) = \frac{1}{c}\frac{S(z,a)}{L(z,a)}.$$

In fact $c = 1$, and here's why. From $|S(z,a)| = |L(z,a)|$ we get $|c| = 1$. Because $S(a,a)$ and $f'(a)$ are real and positive, and because the residue of $L(z,a)$ at a is $1/2\pi$, we see that $c = 1$.

[3] It *is* possible to prove separately a stronger version of Caratheodory's theorem using the techniques Bell develops, and so to give a self-contained treatment of the Riemann mapping theorem (at least for C^∞ boundaries).

What about the extremal property? We want to see that

$$f(z) = \frac{S(z,a)}{L(z,a)}$$

is extremal for the problem

$$\max_{g \in \mathcal{F}} g'(a),$$

where $\mathcal{F}$ is the class of analytic functions g in Ω with $|g| < 1$, $g(a) = 0$ and $g'(a) > 0$. Actually, to make the following argument work we have to assume that the functions g extend at least continuously to $\partial\Omega$ to be able to integrate.

We'll need to know that $f'(a) = 2\pi S(a,a)$. To see this (one way) write

$$S(z,a) = f(z)L(z,a) = ((f'(a)(z-a) + O((z-a)^2) \left(\frac{1}{2\pi}\frac{1}{z-a} + \text{analytic}\right)$$

Next, consider the square of the Garabedian kernel, L_a^2. The function L_a has a simple pole at a with residue $1/2\pi$, and the claim is that L_a^2 has zero residue at a. For this use the fundamental identity between L and S, according to which

$$L_a^2 T = i L_a \overline{S_a}.$$

We can then calculate the residue of L_a^2 by

$$\text{Res}_a L_a^2 = \frac{1}{2\pi i}\int_{\partial\Omega} L_a^2(\zeta)\, d\zeta = \frac{1}{2\pi i}\int_{\partial\Omega} L_a^2 T\, |d\zeta| = \frac{1}{2\pi i}\int_{\partial\Omega} i l_a \overline{S_a}\, |d\zeta| = \frac{1}{2\pi}(L_a, S_a).$$

This is zero because L_a and S_a are orthogonal in $L^2(\partial\Omega)$! Because the residue is zero we can write

$$L_a^2(z) = \frac{1}{4\pi^2}\frac{1}{(z-a)^2} + \text{analytic},$$

and now if g is a competing function for the extremal problem we can calculate $g'(a)$ via

$$\begin{aligned} g'(a) = 4\pi^2 \text{Res}_a(L_a^2 g) &= \frac{2\pi}{i}\int_{\partial\Omega} L_a^2(\zeta) g(\zeta)\, d\zeta \quad (\text{remember, } g(a) = 0) \\ &= \frac{2\pi}{i}\int_{\partial\Omega} L_a^2(\zeta) g(\zeta) T(\zeta)\, |d\zeta|. \end{aligned}$$

We then find that

$$g'(a) = |g'(a)| \le 2\pi \int_{\partial\Omega} |L_a|^2\, |d\zeta|,$$

because $|g| < 1$. Finally, we appeal once more to the identity between S and L, using that $|S_a(\zeta)| = |L_a(\zeta)|$. This gives

$$g'(a) \le 2\pi \int_{\partial\Omega} |S_a|^2\, |d\zeta| = 2\pi(S_a, S_a) = 2\pi S_a(a) = 2\pi S(a,a) = f'(a). \quad \text{QED}$$

Remark This argument is the very essence of 'the method of contour integration': General identities via orthogonality – in this case the fundamental properties of S and L – combined with ingenious, explicit residue calculations.

Ahlfors attended to:

We can now formulate a second version of the Ahlfors mapping theorem.

> **Ahlfors Mapping Theorem, second version** Let Ω be a domain of connectivity n with C^∞ boundary and let $a \in \Omega$. The function
>
> $$f(z) = \frac{S(z,a)}{L(z,a)}$$
>
> is an $n:1$ mapping of Ω onto $\mathbf{D}$.

The final version of the theorem – the one we'll prove – will include the statements that $L(z,a)$ is nonvanishing in $\overline{\Omega}\backslash\{a\}$ and $S(z,a)$ is nonvanishing on $\partial\Omega$ and has exactly $n-1$ zeros in Ω. We'll also discuss boundary behavior and, of course, the fact that $S(z,a)/L(z,a)$ solves the extremal problem.

It's hard not to be impressed with this development. In increasing generality, the functions are

- The Möbius transformation of $\mathbf{D}$ onto itself
- The conformal mapping of a simply connected domain onto $\mathbf{D}$
- The Ahlfors mapping of a multiply connected domain onto $\mathbf{D}$

Each is given by the same formula! They all satisfy the same type of extremal problem, true – but to be given by the same formula?[4] This seems to me to reflect the 'method of functional analysis' in a particularly strong way. The spaces and the operators are (formally) the same in all cases. Nowhere in the set-up do we distinguish the disk from a general simply connected domain, or a simply connected domain from a multiply connected domain.

Boundary values and $\bar{\partial}$

It's time to develop some of the ideas and techniques that support this approach to the mapping theorems. The uninterested reader can skip to the section on the Ahlfors mapping theorem for the thrilling climax.

Much of Bell's book is devoted to real variable methods applied to boundary value problems. The starting point is Pompieu's formula:

> If $u \in C^1(\overline{\Omega})$ then
>
> $$u(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{\Omega} \frac{\bar{\partial} u(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$
>
> for all $z \in \Omega$.

A few comments about this:

- The formula is proved via Green's theorem applied to the function $u(\zeta)/(\zeta - z)$ (z fixed) on the domain Ω minus a small disk centered at z. In other words, it's proved just like the Cauchy integral formula for analytic functions, and it reduces to that when u is analytic, for then $\bar{\partial}u = 0$.

[4] And why the *quotient* of the two kernel functions?

A (complex) form of Green's theorem that's used in this derivation, and that we'll use later, reads

$$\int_{\partial\Omega} u\, d\zeta = \int_{\Omega} \bar{\partial} u\, d\bar{\zeta} \wedge d\zeta.$$

- The second integral – the area integral over Ω – has a kernel with a singularity at z. It's clear, however, that for any fixed $z \in \Omega$ the function $\bar{\partial}u(\zeta)/(\zeta - z)$ is integrable; introduce local polar coordinates centered at z.

- If u vanishes on $\partial\Omega$, for example if u has compact support in Ω, then the first integral is zero and the formula reduces to

$$u(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{\bar{\partial}u(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

  We'll use this integral to *obtain* a solution of the $\bar{\partial}$-equation.

One can relax the smoothness assumptions on u (and on $\partial\Omega$).[5]

Solving the $\bar{\partial}$ - equation

One of the most widely used results in this line of work is an integral formula for solving $\bar{\partial}u = v$.[6] The result is

Theorem [Solving $\bar{\partial}$] Let $v \in C^\infty(\overline{\Omega})$ and define

$$u(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{v(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}, \quad z \in \overline{\Omega}.$$

Then $u \in C^\infty(\overline{\Omega})$ and $\bar{\partial}u = v$ on $\overline{\Omega}$.

The proof of this theorem is a perfect example of 'real variable techniques' in the study of analytic functions. It seems to me that, more often than not, this means the systematic use of:

(a) C^∞ functions with compact support, used as

(b) Smooth cut-off functions.

(c) Maybe even a partition of unity here and there.

We've already seen various spaces of C^∞ functions enter the picture. Cut-offs are soon to follow.

To establish the smoothness up to $\partial\Omega$ as stated in the theorem we'll need the following lemma.

Lemma [Bell's boundary lemma] Let $v \in C^\infty(\overline{\Omega})$. For each positive integer m there is a function $\Phi_m \in C^\infty(\overline{\Omega})$ with $\Phi_m = 0$ on $\partial\Omega$ and $v = \bar{\partial}\Phi_m$ to order m on $\partial\Omega$.

[5] Questions on this point are important for applications of the formula to the study of the Beltrami equation and quasiconformal (qc) mappings; see, for example Lehto and Virtanen's book on qc mappings for a careful exposition.

[6] One reason why it's helpful to solve the $\bar{\partial}$-equation is that it provides a way of obtaining *analytic* functions, for if $\bar{\partial}u_1 = v$ and $\bar{\partial}u_2 = v$ then $u_1 - u_2$ is analytic. (This comes up more in several complex variables and in the theory of complex manifolds where one wants to piece together local analytic functions (germs) to get a global analytic function.) One reason why it's helpful to have an integral formula for the solution is that it can provide bounds on the solution from bounds on v.

Assuming the Lemma, here's how the proof of the theorem goes.

Let's first prove that the integral formula in the theorem gives a solution of $\bar{\partial} u = v$ in Ω. This involves using the Pompieu formula in conjunction with a smooth cut-off function. Fix a point $z_0 \in \Omega$ and let χ be a smooth function that is 1 in a neighborhood N of z_0 and has compact support in Ω. (In particular, χ vanishes on $\partial\Omega$.)

Write $v = (1-\chi)v + \chi v$. Then, by how u is defined,

$$\begin{aligned} 2\pi i u(z) &= \int_\Omega \frac{(1-\chi)v}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} + \int_\Omega \frac{\chi v}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} \\ &= \int_{\Omega \setminus \overline{N}} \frac{(1-\chi)v}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} + \int_\Omega \frac{\chi v}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} \end{aligned}$$

We want to show that $\bar{\partial} u(z_0) = v(z_0)$ by taking $\bar{\partial}$ of the right hand side, and for that we want to differentiate under the integral signs.

Look at the first integral. Near z_0 the integral is clearly an analytic function of z, since ζ varies over $\Omega \setminus \overline{N}$ and so stays away from z. Thus we can take $\bar{\partial}$ by differentiating under the integral sign and this gives 0.

Look at the second integral. Since χ has compact support we can write the integral over $\mathbf{C}$ and change variables:

$$\int_\Omega \frac{(\chi v)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = \int_{\mathbf{C}} \frac{(\chi v)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = \int_{\mathbf{C}} (\chi v)(\xi + z)\frac{1}{\xi}\, d\xi \wedge d\bar{\xi}.$$

Now, $1/\xi$ is locally integrable and χv is smooth of compact support, so we can differentiate under the integral (putting the derivative on χv) and then make a change of variables back, *i.e.*,

$$\bar{\partial}\left(\int_{\mathbf{C}} (\chi v)(\xi + z)\frac{1}{\xi}\, d\xi \wedge d\bar{\xi}\right) = \int_{\mathbf{C}} \bar{\partial}(\chi v)(\xi + z)\frac{1}{\xi}\, d\xi \wedge d\bar{\xi} = \int_\Omega \frac{\bar{\partial}(\chi v)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

But let's now apply Pompieu's formula. The boundary term vanishes because χ has compact support in Ω, and we are left with

$$\frac{1}{2\pi i}\int_\Omega \frac{\bar{\partial}(\chi v)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = (\chi v)(z).$$

Combining this with the first part of the proof (where z is near z_0) we then have

$$\bar{\partial} u(z_0) = (\chi v)(z_0) = v(z_0).$$

Since $z_0 \in \Omega$ was arbitrary we conclude that $\bar{\partial} u = v$ in Ω. This also shows that u (as defined by the integral) is C^∞ in Ω (because v is C^∞). It *doesn't* show that u extends to be C^∞ on the boundary. That requires a separate argument using Bell's boundary lemma, above, which we'll do now. Once we know this it follows that $\bar{\partial} u = v$ on $\overline{\Omega}$.

We'll show that u is of class C^m on $\overline{\Omega}$ for each positive integer m. Given m, let Φ be the function furnished by the lemma. Then $\Psi = v - \bar{\partial}\Phi$ vanishes to order m on $\partial\Omega$, and hence may be considered a C^m function on all of $\mathbf{C}$ by setting it equal to zero outside Ω. Because Φ vanishes on $\partial\Omega$, Pompieu's formula gives

$$\Phi(z) = \frac{1}{2\pi i}\int_\Omega \frac{\bar{\partial}\Phi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}, \quad z \in \Omega,$$

and using

$$u(z) = \frac{1}{2\pi i}\int_\Omega \frac{v(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta,$$

we obtain

$$u(z) - \Phi(z) = \frac{1}{2\pi i}\int_\Omega \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta.$$

Change variables in the integral as in the earlier part of the proof:

$$u(z) - \Phi(z) = \frac{1}{2\pi i}\int_C \Psi(\xi + z)\frac{1}{\xi}\, d\xi \wedge d\bar\xi.$$

The right hand side makes sense for $z \in \mathbf{C}$ (Ψ is of compact support.) This makes it clear that $u - \Phi$ is C^m in $\overline{\Omega}$ (the integral can be differentiated m times), and hence $u \in C^m(\overline{\Omega})$

This completes the proof of the theorem. We now turn to the proof of the lemma for more cut-off functions in action.

‘Defining functions’ for a domain and Bell’s Boundary Lemma

A very useful technical device is a *defining function* for a domain. (This is used all the time in several complex variables, less so in the plane.) A domain Ω of the type we’re considering can be specified by a function ρ, C^∞ on a neighborhood of $\overline{\Omega}$, through the conditions:

(a) $\Omega = \{z \in \mathbf{C}\colon \rho(z) < 0\}$

(b) $\partial\Omega = \{z \in \mathbf{C}\colon \rho(z) = 0\}$

(c) $d\rho \neq 0$ on $\partial\Omega$

I won’t go through he construction of such a ρ: use local diffeomorphisms of neighborhoods that cover $\partial\Omega$ to a neighborhood of the real axis, pull back $\operatorname{Im} z$ and use a partition of unity to piece these maps together to define ρ.

Among other things, the use of a defining function allows a useful characterization of what it means for a function to vanish to a given order on the boundary. Namely:

A function v vanishes to order m on $\partial\Omega$ if and only if $v = \theta\rho^{m+1}$ for some $\theta \in C^\infty(\overline{\Omega})$.

Prove this locally using Taylor series with remainder, first for the lower half-plane with $\rho(z) = -\operatorname{Im} z$, and then piece together the results with a partition of unity.

Proof of Bell’s Boundary Lemma The proof goes by induction on m. Take the case $m = 0$. We want to produce a function Φ_0 in $C^\infty(\overline{\Omega})$ that vanishes on $\partial\Omega$ and has $v = \bar\partial\Phi_0$ on $\partial\Omega$. As noted above, we can write $\Phi_0 = \theta_0\rho$ for some θ_0, and we ask if we can choose θ_0 appropriately. We have

$$\bar\partial\Phi_0 = (\bar\partial\theta_0)\rho + \theta_0\bar\partial\rho.$$

If we put

$$\theta_0 = \frac{v}{\bar\partial\rho}$$

then

$$\bar\partial\Phi_0 = (\bar\partial\theta_0)\rho + v \quad \text{on } \Omega \text{ and hence} \quad \bar\partial\Phi_0 = v \quad \text{on } \partial\Omega.$$

The problem with this is the (possible) points in Ω where $\bar\partial\rho$ vanishes. (Note that $d\rho \neq 0$ on $\partial\Omega$ so $\bar\partial\rho \neq 0$ on $\partial\Omega$.)

To fix this, let χ be a C^∞ function on $\mathbf{C}$ that is 1 in a neighborhood of $\partial\Omega$ and vanishes on a neighborhood of $\{\bar\partial\rho = 0\}$, and set

$$\Phi_0 = \chi\frac{v}{\bar\partial\rho}.$$

This does the trick.

Now suppose that the lemma has been proved for all $k < m$. Then there is a function Φ_{m-1} in $C^\infty(\overline{\Omega})$, vanishing on $\partial\Omega$ such that $\bar\partial\Phi_{m-1} - v$ vanishes to order $m-1$ on $\partial\Omega$. That is

$$\bar\partial\Phi_{m-1} - v = \Psi_{m-1}\rho^m$$

We'll set

$$\Phi_m = \Phi_{m-1} - \theta_m\rho^{m+1}$$

for a well chosen θ_m. First, we have

$$\begin{aligned}\bar\partial\Phi_m &= \bar\Phi_{m-1} - \bar\partial(\theta_m\rho^{m+1}) \\ &= v + \Psi_{m-1}\rho^m - (\bar\partial\theta_m)\rho^{m+1} - \theta_m(m+1)\rho^m\bar\partial\rho\end{aligned}$$

The term with the ρ^{m+1} vanishes to order m so we're in good shape with that one. We want to choose θ_m so that the remaining two terms cancel. This is easy to do with

$$\theta_m = \frac{1}{m+1}\frac{\Psi_{m-1}}{\bar\partial\rho},$$

using again the cut-off function χ. This completes the induction and the proof of the lemma.

The Cauchy Transform and its Adjoint

Let $u \in C^\infty(\partial\Omega)$. Recall that the Cauchy transform of u is the analytic function on Ω defined by

$$(\mathcal{C}\,u)(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{u(\zeta)}{\zeta - z}\,d\zeta.$$

The theorem on solving $\bar\partial$, in particular the statement on the smoothness of the solution, allows us to easily deduce the following.

Theorem [Smoothness of $\mathcal{C}\,u$] $\mathcal{C}\,u$ is analytic in Ω and C^∞ on $\overline{\Omega}$.

Proof: It's clear that $\mathcal{C}\,u$ is analytic in Ω. The real result here (a small one, but you've got to start by saying something) is the smoothness on $\partial\Omega$. For that, let U be any C^∞ function in $\overline{\Omega}$ that is equal to u on $\partial\Omega$. Using Pompieu's formula we can write

$$\begin{aligned}U(z) &= \frac{1}{2\pi i}\int_{\partial\Omega}\frac{U(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\Omega}\frac{\bar\partial U(\zeta)}{\zeta - z}\,d\zeta\wedge d\bar\zeta \\ &= (\mathcal{C}\,u)(z) + \frac{1}{2\pi i}\int_{\Omega}\frac{\bar\partial U(\zeta)}{\zeta - z}\,d\zeta\wedge d\bar\zeta.\end{aligned}$$

The theorem on solving $\bar\partial$ implies that the function defined by the area integral is in $C^\infty(\overline{\Omega})$. Hence so is $\mathcal{C}\,u$ and the theorem is proved.

This theorem can be restated as saying that $\mathcal{C}$ maps $C^\infty(\partial\Omega)$ into $A^\infty(\Omega)$. What about the boundary values of $\mathcal{C}\,u$? One formulation of an answer is given by Bell as:

Theorem [Boundary values of $\mathcal{C}\,u$] Let $u \in C^\infty(\partial\Omega)$ and let m be a positive integer. There is a function $\Psi \in C^\infty(\overline{\Omega})$ that vanishes to order m on $\partial\Omega$ such the boundary values of $\mathcal{C}\,u$ are given by

$$(\mathcal{C}\,u)(z) = u(z) - \frac{1}{2\pi i}\int_\Omega \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar{\zeta}, \quad z \in \partial\Omega.$$

To prove this, let U be a function in $C^\infty(\overline{\Omega})$ that agrees with u on $\partial\Omega$, and let Φ be a function provided by Bells's boundary lemma with $\Phi = 0$ on $\partial\Omega$ and $\bar{\partial}U = \bar{\partial}\Phi$ to order m on $\partial\Omega$. By Pompieu's formula

$$\begin{aligned} U(z) - \Phi(z) &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{U(\zeta) - \Phi(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_\Omega \frac{\bar{\partial}U - \bar{\partial}\Phi}{\zeta - z}\,d\zeta \wedge d\bar{\zeta} \\ &= (\mathcal{C}\,u)(z) + \frac{1}{2\pi i}\int_\Omega \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar{\zeta}, \end{aligned}$$

where

$$\Psi = \bar{\partial}U - \bar{\partial}\Phi.$$

As before, we can regard Ψ as being a function in $C^\infty(\mathbf{C})$ by setting it equal to 0 off Ω, and then by changing variables and differentiating under the integral sign it follows that $\mathcal{C}\,u$ extends smoothly to $\partial\Omega$. Since $U = u$ and $\Phi = 0$ on $\partial\Omega$ the boundary values of $\mathcal{C}\,u$ can then be expressed as

$$(\mathcal{C}\,u)(z) = u(z) - \frac{1}{2\pi i}\int_\Omega \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar{\zeta}, \quad z \in \partial\Omega.$$

With this result we can find a formula for the (formal) adjoint of $\mathcal{C}$. For the following calculation it will be convenient to use the notation

$$\mathcal{I}(z) = \frac{1}{2\pi i}\int_\Omega \frac{\Psi(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar{\zeta},$$

where Ψ is the function in the preceding theorem. Thus

$$\mathcal{C}\,u = u - \mathcal{I}.$$

Now, let $u, v \in C^\infty(\partial\Omega)$ and let T be the unit tangent vector field along $\partial\Omega$, so that $|dz| = \overline{T}dz$. With $(\mathcal{C}\,u, v) = (u - \mathcal{I}, v)$ we compute the term $(\mathcal{I}, v)$ using Fubini's theorem (justification suppressed):

$$\begin{aligned} &\int_{\partial\Omega} \left(\frac{1}{2\pi i}\frac{\Psi(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar{\zeta}\right)\overline{v}(z)\,|dz| \\ &= \int_\Omega \Psi(\zeta)\left(\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\overline{v(z)}}{\zeta - z}\,|dz|\right)\,d\zeta \wedge d\bar{\zeta} \\ &= \int_\Omega \Psi(\zeta)\left(-\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\overline{v(z)T(z)}}{z - \zeta}\,dz\right)\,d\zeta \wedge d\bar{\zeta} \\ &= \int_\Omega \Psi(\zeta)\,\mathcal{C}(-\overline{v}\overline{T})(\zeta)\,d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

Recall that $\Psi = \bar{\partial}(U - \Phi)$ where Φ vanishes on $\partial\Omega$ and Ψ vanishes to order m on $\partial\Omega$. Furthermore, $\mathcal{C}(-\bar{v}\bar{T})$ is analytic in Ω so its $\bar{\partial}$-derivative is zero there. The last integral can then be written

$$\begin{aligned}\int_\Omega \Psi(\zeta)\,\mathcal{C}(-\bar{v}\bar{T})(\zeta)\,d\zeta \wedge d\bar{\zeta} &= \int_\Omega \bar{\partial}(U-\Phi)\,\mathcal{C}(-\bar{v}\bar{T})\,d\zeta \wedge d\bar{\zeta} \\ &= \int_\Omega \bar{\partial}((U-\Phi)\,\mathcal{C}(-\bar{v}\bar{T}))\,d\zeta \wedge d\bar{\zeta}\end{aligned}$$

This is all set up to apply the complex form of Green's theorem:

$$\int_\Omega \bar{\partial}((U-\Phi)\,\mathcal{C}(-\bar{v}\bar{T}))\,(-d\bar{\zeta}\wedge d\zeta) = \int_{\partial\Omega}(U-\Phi)\,\mathcal{C}(\bar{v}\bar{T})\,d\zeta.$$

But $\Phi = 0$ and $U = u$ on $\partial\Omega$, so

$$\int_{\partial\Omega}(U-\Phi)\,\mathcal{C}(\bar{v}\bar{T})\,d\zeta = \int_{\partial\Omega} u\,\mathcal{C}(\bar{v}\bar{T})\,d\zeta = \int_{\partial\Omega} u\,\mathcal{C}(\bar{v}\bar{T})T|d\zeta| = (u, \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T})$$

Therefore

$$(\mathcal{C}\,u, v) = (u - \mathcal{I}, v) = (u, v) - (\mathcal{I}, v) = (u, v) - (u, \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T}) = (u, v - \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T})$$

In other words

$$(\mathcal{C}\,u, v) = (u, \mathcal{C}^* v),$$

where

$$\mathcal{C}^* v = v - \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T}$$

This defines an adjoint of $\mathcal{C}$, but only on $C^\infty(\partial\Omega)$. We want to extend the definition to $L^2(\partial\Omega)$. For this, and for other properties of the Cauchy transform, we need a relationship established by Kerzman and Stein between the Szegö projection, the Cauchy transform, and its adjoint.

The Kerzman-Stein formula Let $u \in C^\infty(\partial\Omega)$. From the formula for $\mathcal{C}^*$ we have

$$u + \mathcal{C}\,u - \mathcal{C}^* u = u + \mathcal{C}\,u - (u - \overline{\mathcal{C}(\bar{u}\bar{T})}\bar{T}) = \mathcal{C}\,u + \overline{\mathcal{C}(\bar{u}\bar{T})}\bar{T}.$$

Look at this equation in terms of the orthogonal decomposition $L^2(\partial\Omega) = H^2(\partial\Omega) \oplus H^2(\partial\Omega)^\perp$, and, concretely, in terms of the expression $h + \overline{TH}$ in components relative to that decomposition. The term $\mathcal{C}\,u$ is in $A^\infty(\Omega)$, and hence in $H^2(\partial\Omega)$. The term $\overline{\mathcal{C}(\bar{u}\bar{T})}\bar{T}$ is of the form $\overline{TH}$ with $H = \mathcal{C}(\bar{u}\bar{T}) \in A^\infty(\Omega)$ – thus $\overline{\mathcal{C}(\bar{u}\bar{T})}\bar{T} \in H^2(\partial\Omega)^\perp$. Perfect. In terms of the Szegö projection $P\colon L^2(\partial\Omega) \to H^2(\partial\Omega)$ we may therefore write

$$P(I + (\mathcal{C} - \mathcal{C}^*))u = \mathcal{C}\,u$$

for $u \in C^\infty(\partial\Omega)$, where I is the identity. This is the *Kerzman-Stein identity*. We want to see that it holds on $L^2(\partial\Omega)$.

Introduce the operator

$$\mathcal{A} = \mathcal{C} - \mathcal{C}^*.$$

Currently this is defined only on $C^\infty(\partial\Omega)$. However, as we indicated earlier, Kerzman and Stein show that $\mathcal{A}$ can be given as an integral operator,

$$(\mathcal{A}u)(z) = \int_{\partial\Omega} A(z,\zeta)u(\zeta)\,d\zeta, \quad z \in \partial\Omega,$$

where $A(z, \zeta)$ is C^∞ as a function of (z, ζ) on $\partial\Omega \times \partial\Omega$. (Bell's proof of this differs from Kerzman and Stein's. We won't go through his arguments – this is where one uses the Plemelj formula for boundary values of the Cauchy integral in terms of the Hilbert transform.) As a consequence of this, $\mathcal{A}$ is defined on $L^2(\partial\Omega)$, maps $L^2(\partial\Omega)$ into $C^\infty(\partial\Omega)$, and satisfies an L^2 estimate,

$$||\mathcal{A}|| \leq c||u||.$$

Write the Kerzman-Stein formula as

$$P(I + \mathcal{A}) = \mathcal{C}$$

on $C^\infty(\partial\Omega)$. Now $||P|| = 1$ on $L^2(\partial\Omega)$, because it's a projection, and on $C^\infty(\partial\Omega)$ we conclude from the bound on $\mathcal{A}$ that

$$||\,\mathcal{C}\,u|| \leq (1 + c)||u||.$$

Since $C^\infty(\partial\Omega)$ is dense in $L^2(\partial\Omega)$ and $\mathcal{C}$ maps $C^\infty(\partial\Omega)$ into $A^\infty(\Omega)$ it now follows that $\mathcal{C}$ has an extension to a bounded operator on $L^2(\partial\Omega)$. (We continue to denote the extension by $\mathcal{C}$, of course.) Because of this, we can use the formula

$$\mathcal{C}^* v = v - \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T}$$

to extend $\mathcal{C}^*$ to a bounded operator on $L^2(\partial\Omega)$ (same symbol) and it follows from the identity

$$(\mathcal{C}\,u, v) = (u, \mathcal{C}^* v)$$

on $C^\infty(\partial\Omega)$ that the extension is the L^2-adjoint of $\mathcal{C}$. Finally, the Kerzman-Stein formula

$$P(I + \mathcal{A}) = P(I + (\mathcal{C} - \mathcal{C}^*)) = \mathcal{C}$$

holds on $L^2(\partial\Omega)$, and hence $\mathcal{C}$ maps $L^2(\partial\Omega)$ into $H^2(\partial\Omega)$.

Taking the L^2-adjoint of the Kerzman-Stein formula, and a little algebraic fooling around, leads to the identity

$$P = \mathcal{C} - \mathcal{A}(I - P).$$

There's a consequence of this: $\mathcal{A}$ maps $L^2(\partial\Omega)$ into $C^\infty(\partial\Omega)$ and hence so does $\mathcal{A}(I - P)$. Furthermore, both $\mathcal{A}(I - P)$ and $\mathcal{C}$ preserve the space $C^\infty(\partial\Omega)$. We deduce a result promised earlier, that the Szegö projection P maps $C^\infty(\partial\Omega)$ into itself.

Let's summarize the main results of this section

> **Theorem** The Cauchy transform has a bounded extension mapping $L^2(\partial\Omega)$ into $H^2(\partial\Omega)$. The Szegö projection $P\colon L^2(\partial\Omega) \to H^2(\partial\Omega)$ maps $C^\infty(\partial\Omega)$ into itself. On $L^2(\partial\Omega)$ we have the Kerzman-Stein identity, $P(I + \mathcal{A}) = \mathcal{C}$.

The Orthogonal Decomposition of $L^2(\partial\Omega)$ Recall that we write

$$L^2(\partial\Omega) = H^2(\partial\Omega) \oplus H^2(\partial\Omega)^\perp,$$

with the Szegö projection $P\colon L^2(\partial\Omega) \to H^2(\partial\Omega)$ and Garabedian projection $P^\perp\colon L^2(\partial\Omega) \to H^2(\partial\Omega)^\perp$.

We showed earlier that a function of the form $\overline{TH}$ with $H \in H^2(\partial\Omega)$ is orthogonal to any $h \in H^2(\partial\Omega)$. Let us now show conversely that if $u \in L^2(\partial\Omega)$ then $v = u - Pu = (I - P)u \in H^2(\partial\Omega)^\perp$ is of the form $\overline{TH}$.

Since $\mathcal{C}$ maps $L^2(\partial\Omega)$ into $H^2(\partial\Omega)$

$$0 = (\mathcal{C}\, u, v) = (u, \mathcal{C}^* v),$$

and as this holds for all $u \in L^2(\partial\Omega)$ it must be that $\mathcal{C}^* v = 0$. Using the formula for the adjoint,

$$0 = \mathcal{C}^* v = v - \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T}, \quad \text{i.e.} \quad v = \overline{\mathcal{C}(\bar{v}\bar{T})}\bar{T}$$

This proves the result with $H = \mathcal{C}(\bar{v}\bar{T}) \in H^2(\partial\Omega)$.

Finally, we want to show that if $u \in L^2(\partial\Omega)$ then, actually, $H = P(\overline{uT})$. For this, write $u = h + \overline{HT}$ and multiply by T:

$$Tu = Th + \overline{H}.$$

Now take complex conjugates:

$$\overline{Tu} = \overline{Th} + H.$$

This is the orthogonal decomposition for $\overline{Tu}$ where H is the part in $H^2(\partial\Omega)$, thus $H = P(\overline{Tu})$.

The Ahlfors Mapping Theorem

Here's a complete statement of the Ahlfors mapping theorem, as formulated in Bell's book.

> **Ahlfors Mapping Theorem** Let Ω be a bounded domain of connectivity $n \geq 1$ with C^∞ smooth boundary, and let $a \in \Omega$. The Garabedian kernel $L_a(z)$ is nonvanishing for $z \in \Omega \backslash \{a\}$. The Szegö kernel $S_a(z)$ is nonvanishing on $\partial\Omega$ and has exactly $n-1$ zeros in Ω. The function
>
> $$f(z) = \frac{S_a(z)}{L_a(z)} = \frac{S(z,a)}{L(z,a)}$$
>
> maps Ω onto the unit disk $\mathbf{D}$ in an $n{:}1$ manner, counting multiplicities. f extends to be C^∞ on $\overline{\Omega}$, f' is nonvanishing on the boundary and f maps each boundary curve one-to-one onto the unit circle. Among all analytic functions g in Ω with values in $\mathbf{D}$ the functions that maximize $|g'(a)|$ are of the form $e^{i\theta} f(z)$. f is the unique function in the class with $f'(a) > 0$.

There's less gnashing of teeth if we assume at first that $\partial\Omega$ is (real) analytic; we'll remove this restriction at the end of the proof.

Real analytic boundaries A few details on this. Bell expresses the needed results as:

> **Theorem** Let Ω be a bounded domain with real-analytic boundary. The Cauchy transform maps $C^\omega(\partial\Omega)$ into itself and so too does the Szegö projection.

The key thing to know is that when the boundary is analytic the Kerzman-Stein kernel $A(z,\zeta)$ is a real analytic function of $(z,\zeta) \in \partial\Omega \times \partial\Omega$. This is easy to show, given the earlier arguments (which we haven't given). From this it follows that S_a and L_a extend to be analytic past $\partial\Omega$. For the proof of Ahlfors's theorem we then know that f is meromorphic on $\overline{\Omega}$; it will emerge that f is actually analytic.

Extended argument principle One benefit of moving meromorphically beyond $\partial\Omega$ is that we are then justified in applying the argument principle to $\overline{\Omega}$ *including* the possibility that there are zeros and poles on $\partial\Omega$. We recall the extended form of the argument principle, where zeros and poles on the boundary are counted with half their multiplicities:

$$\begin{aligned}\frac{1}{2\pi}\Delta_{\partial\Omega}\arg g &= \text{Number of zeros of } g \text{ in } \Omega - \text{Number of poles of } g \text{ in } \Omega \\ &+ \frac{1}{2}\text{Number of zeros of } g \text{ on } \partial\Omega - \frac{1}{2}\text{Number of poles of } g \text{ on } \partial\Omega\end{aligned}$$

Counting hits: It's hip to be a square Right now we know that f has a zero at $z = a$ because $S(a,a) > 0$ and $L(z,a)$ has a simple pole at $z = a$. The other thing we know is the boundary behavior of S and L, *i.e.*, we know the identity relating the two kernels on the boundary. To bring this to bear on the mapping properties of f we consider a complex number λ with $|\lambda| = 1$ and ask how many times $f(z) = \lambda$ in $\overline{\Omega}$. Now, f is supposed to map $\partial\Omega$ to $\partial\mathbf{D}$, so

$$f(z) = \frac{S(z,a)}{L(z,a)} = \lambda$$

should only have solutions – and precisely n solutions – when $z \in \partial\Omega$. Proving this is where the extended argument principle and the identity

$$S(a,z) = \frac{1}{i}L(z,a)T(z), \quad z \in \partial\Omega.$$

come in.

Consider the function

$$G(z) = S(z,a) - \lambda L(z,a) \quad \text{for} \quad z \in \overline{\Omega}.$$

We want to show that G has n zeros in $\overline{\Omega}$, exactly one on each component of $\partial\Omega$.

Let $z \in \partial\Omega$, multiply the equation defining G by T, and use the identity for $S(a,z)$ and $L(z,a)$ together with the Hermitian symmetry of S and $T\overline{T} = 1$:

$$\begin{aligned}G(z)T(z) &= S(z,a)T(z) - \lambda L(z,a)T(z) \\ &= \overline{S(a,z}T(z) - \lambda i S(a,z) \\ &= i\overline{L(z,a)} - \lambda i\overline{S(z,a)} \\ &= i\lambda(\bar{\lambda}\overline{L(z,a)} - \overline{S(z,a)}) \quad (\lambda\bar{\lambda} = 1) \\ &= -i\lambda\overline{G(z)}\end{aligned}$$

Now multiply through by $G\overline{T}$. This gives

$$G^2(z) = -i\lambda|G(z)|^2\overline{T(z)}, \quad z \in \partial\Omega.$$

A square on the left-hand side... An innocent observation? Hardly. Watch this.

Let $\gamma_1, \ldots, \gamma_n$ be the boundary curves of Ω; these are real-analytic, simple closed curves. Suppose G does not vanish on a γ_i. Then the change in the argument of $G^2(z)$ along γ_i must be either 0 or an *even* multiple of $\pm 2\pi$. But the change in the argument of $T(z)$ along γ_i is $\pm 2\pi$, and up to sign this is also then the change in the argument of $-i\lambda|G(z)|^2\overline{T(z)}$ along γ_i. Thus the equality above leads to a contradiction unless G vanishes at least once on γ_i, *i.e.*, we conclude that G has at least one zero on each boundary curve of Ω.

Now let's apply the extended argument principle to G on $\overline{\Omega}$. We know that G has a single simple pole at a. Thus, on the one hand,

$$\frac{1}{2\pi}\Delta \arg_{\partial\Omega} G = \text{Number of zeros of } G \text{ in } \Omega + \frac{1}{2}\text{Number of zeros of } G \text{ on } \partial\Omega - 1.$$

On the other hand

$$\frac{1}{2\pi}\Delta \arg_{\partial\Omega} G^2 = \frac{1}{2\pi}\Delta_{\partial\Omega} \arg \overline{T} = -1 + (n-1) = n-2.$$

(The 'outermost' boundary curve has T going around counterclockwise and the remaining $n-1$ boundary curves have T going around clockwise; this is reversed for $\overline{T}$.) Thus

$$\frac{1}{2\pi}\Delta \arg_{\partial\Omega} G = \frac{n-2}{2} = \frac{n}{2} - 1.$$

But

$$\text{Number of zeros of } G \text{ on } \partial\Omega \geq n$$

since, as we have just seen, G must vanish *at least once* on each boundary curve. Hence

$$\text{Number of zeros of } G \text{ in } \Omega + \frac{n}{2} - 1 \leq \frac{1}{2\pi}\Delta \arg_{\partial\Omega} G = \frac{n}{2} - 1.$$

We can now say that G has *no* zeros in Ω and *exactly one* zero on each boundary curve.[7]

What about f on $\partial\Omega$? First, we claim that f has no poles in $\overline{\Omega}$. If it did then there would be a point $p \in \overline{\Omega}$ near the pole where $|f(p)| > 1$ and $L(p,a) \neq 0$. Join that point to a along a curve in Ω that doesn't pass through a zero of L_a. Along that curves $|f|$ goes from a value greater than 1 to 0, so it must be 1 somewhere (at a point where $L_a \neq 0$). But $|f|$ is 1 where G is zero, and G cannot vanish at a point in the interior of Ω. Thus f can have no poles in $\overline{\Omega}$.

In fact, this reasoning shows that $|f| \leq 1$ on $\overline{\Omega}$, and since $f(a) = 0$ the maximum principle implies that $|f| < 1$ on Ω. From the identity $iS(a,z) = L(z,a)T(z)$ on the boundary we see that $|f| = 1$ on the dense set of points on $\partial\Omega$ where $L_a \neq 0$, hence $|f| = 1$ everywhere on $\partial\Omega$.

Actually, $L_a \neq 0$ anyway We'll now show that $L_a \neq 0$ on $\overline{\Omega}\backslash\{a\}$. Among other things, this turns out to be important when basing some potential theory in multiply connected domains on the kernel functions, *i.e.*, solving the Dirichlet problem. (Bell goes through this, too.)

From $f = S_a/L_a$ and knowing that f is analytic on $\overline{\Omega}$, it follows that S_a must vanish wherever L_a does. From $G = S_a - \lambda L_a$, $|\lambda| = 1$, we see that G and S_a likewise vanish wherever L_a does, and since G_a cannot vanish at a point in Ω neither can L_a.

To see that L_a does not vanish anywhere on $\partial\Omega$, suppose by way of contradiction that $L_a(z_0) = 0$ for some $z_0 \in \partial\Omega$. Then $S_a(z_0) = 0$, and so $G(z_0) = 0$ for *any* λ of modulus 1. Since S_a is not identically 0 in Ω, there is a point ξ_0 in the *same* boundary curve as z_0 where $S_a(\xi_0) \neq 0$, and $\xi_0 \neq z_0$. Now $|S_a(\xi_0)| = |L_a(\xi_0)|$, by the fundamental identity, so with $\lambda_0 = S_a(\xi_0)/L_(\xi_0)$, $G = S_a - \lambda_0 L_a$ vanishes at ξ_0. This says that G vanishes at two points on a single boundary curve – a contradiction. Therefore $L_a \neq 0$ on $\overline{\Omega}\backslash\{a\}$.

[7] This part of the proof combines two of my favorite mathematical 'secrets of the universe': (a) It's always useful to have two different ways to calculate the same thing; in this case the identity between $|G|^2$ and $\overline{T}$ gives a relation between arguments. (b) Never underestimate a theorem that counts something; in this case, as in so many cases in complex analysis, that's the argument principle.

Winding things up It follows from the result of the previous paragraph and $|S_a| = |L_a|$ on $\partial\Omega$ that $S_a \neq 0$ on $\partial\Omega$. Once more using $iS(a,z) = L(z,a)T(z)$, $z \in \partial\Omega$, we have

$$\begin{aligned} -\frac{1}{2\pi}\Delta_{\partial\Omega} \arg S_a &= \frac{1}{2\pi}\Delta_{\partial\Omega} \arg L_a + \frac{1}{2\pi}\Delta_{\partial\Omega} \arg T \quad (S(z,a) = \overline{S(a,z)}) \\ &= -1 + 1 - (n-1) = -(n-1) \quad (L_a \text{ has a simple pole at } a). \end{aligned}$$

Thus

$$\frac{1}{2\pi}\Delta_{\partial\Omega} \arg S_a = n - 1$$

and we deduce that S_a has exactly $n-1$ zeros in Ω. Because L_a has a simple pole at a, $f = S_a/L_a$ has exactly n zeros in Ω.

For any $w \in \mathbf{D}$ the number of times f assumes w in Ω is given by

$$M(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta) - w}\, d\zeta.$$

Since $M(0) = n$ and $M(w)$ an analytic function on $\mathbf{D}$ taking integer values it must be that $M(w) \equiv n$.

Finally, since $L_a \neq 0$ and $G = S_a - \lambda L_a$ vanishes exactly once on each boundary curve in $\partial\Omega$ for any $|\lambda| = 1$ it follows that f maps each boundary curve one-to-one onto $\partial\mathbf{D}$. This also implies that $f' \neq 0$ on $\partial\Omega$.

The extremal property of f We did this! It is *exactly* the argument we gave for the extremal property of $f = S_a/L_a$ for the Riemann mapping in the simply connected case. The uniqueness of the extremal also follows from this argument; I won't give the details.

The case of C^∞ boundary To drop the assumption that $\partial\Omega$ is analytic we need a transformation formula for the kernel functions under conformal mapping. This is important to know, but I'll be content with recording the answer and referring to Bell (Section 12: also derived later in the book by means of expanding in an orthonormal basis – the usual method.) Here's a full statement.

> **Theorem [Change in S and L under conformal mappings]** Let $f\colon \Omega \to \Omega'$ be a conformal mapping between domains with C^∞ boundaries. Then $f \in C^\infty(\overline{\Omega})$ and $f' \neq 0$ on $\overline{\Omega}$. Consequently $f^{-1} \in C^\infty(\overline{\Omega'})$. Furthermore, f' is a square of a function in $A^\infty(\Omega)$. The kernel functions transform according to
>
> $$\begin{aligned} S_\Omega(z,w) &= \sqrt{f'(z)} S_{\Omega'}(f(z), f(w)) \sqrt{f'(w)} \\ L_\Omega(z,w) &= \sqrt{f'(z)} L_{\Omega'}(f(z), f(w)) \sqrt{f'(w)} \end{aligned}$$

Returning to the present considerations, suppose Ω has a C^∞ smooth boundary. There is a conformal mapping F of Ω onto a domain Ω' with real-analytic boundary. This is a standard construction (which depends on the Riemann mapping theorem) and I won't give the argument. Given a point $a \in \Omega$ it's obvious that the solution of the extremal problem in Ω is $f = e^{i\theta}(\tilde{f} \circ F)$, where $\tilde{f}$ is the extremal function at $F(a)$ for Ω' with $\tilde{f}'(F(a) > 0$ and θ is chosen to make $f'(a) > 0$. In Ω' we know that

$$\tilde{f}(w) = \frac{S_{\Omega'}(w, F(a))}{L_{\Omega'}(w, F(a))},$$

and the transformation formulas then give

$$f(z) = \frac{S_\Omega(z,a)}{L_\Omega(z,a)}$$

as the extremal in Ω.