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Pole Assignment of Linear Uncertain Systems in a Sector Via a Lyapunov-Type Approach

D. Arzelier, J. Bernussou, and G. Garcia

Abstract—**The problem of designing robust control laws, in performance and in stability, for uncertain linear systems is considered here. Performances are taken into account via root clustering of the closed-loop dynamic matrix in a sector of the complex plane. A synthesis procedure, based on a sufficient condition for quadratic stabilizability and root clustering, i.e., γ-stabilizability, is given via the way of an auxiliary convex problem. The results are illustrated by a significant example from the literature.**

I. Introduction

Recently, the issue of guaranteeing stability of uncertain linear systems, while keeping a given level of performance has received increasing interest in the literature. One way to take performances into account, is to specify the location of the poles of the closed-loop system. This approach originates in the general theory of matrix root clustering (see [7], [13]), which mainly allows to state algebraic tests for the spectrum of a given matrix A to be clustered in a wide class of subregions of the complex plane.

Although attempts have been made to develop generalized Lyapunov matrix equations, [13], the link with control theory and control synthesis was not clearly established. More recently, new results on the subject, including robustness concerns for some, were given in [9]–[11]. These last ones consist, for the main part, in stability results and pole assignment criteria and extend some previous results of [15], [18], where quantitative robustness measures and stability margins in the uncertainty parameter space were derived with the help of Lyapunov functions.

Our work is supported by quadratic stabilizability concept. This concept has proven to be very useful in the context of control laws design for uncertain linear systems. Indeed, the foremost works of Meilakhs [14] and Barmish [1], [8], gave rise to numerous developments and particularly to the formulation of the problem of stabilization of uncertain linear systems as a joint search problem for a quadratic Lyapunov function and for a single feedback gain [2], [3], [6]. Here, from a sufficient condition for root clustering in a γ subregion of the complex plane, a linear programming optimization problem is defined. Its solution provides a stabilizing feedback gain together with a quadratic Lyapunov function for uncertain systems over convex parametric uncertainty domain. The advantage of this approach is twofold: first, the test for γ-stabilizability is straightforwardly expressed in terms of a Lyapunov equation; then, it constitutes a real robust control synthesis with its associated numerical procedure, as compared to [9]–[11], which are in fact relative stability tests. An example is developed at the end of the note, borrowed from the literature [16].

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II. Problem Statement

Consider the linear uncertain system governed by the differential equation:

$$\dot{x}(t) = A(r).x(t) + B(s).u(t) \tag{1}$$

where

$x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$, the control. The matrices A, B are of appropriate dimensions.

$$\mathscr{R} = \left\{ r \in \Re^r \middle| \underline{r_k} \le r_k \le \overline{r_k}, \left(\overline{r_k}, \underline{r_k}, \text{ given constants}\right)\right\}$$

$$\mathscr{S} = \left\{ s \in \Re^s \middle| \underline{s_l} \le s_l \le \overline{s_l}, \left(\overline{s_l}, \underline{s_l}, \text{ given constants}\right)\right\}.$$

In addition, we make the assumption that A and B are multilinear functions of their respective argument r and s.

As a consequence, the dynamic matrix A and input matrix B belong, respectively, to the following convex and bounded domains:

$$\mathscr{D}_A = \left\{ A \in \Re^{n.n}: A = \sum_{i=1}^{N} \xi_i.A_i, \qquad \sum_{i=1}^{N} \xi_i = 1, \xi_i \ge 0 \right\} \tag{2}$$

$$\mathscr{D}_B = \left\{ B \in \Re^{n.m}: B = \sum_{j=1}^{M} \beta_j.B_j, \qquad \sum_{j=1}^{M} \beta_j = 1, \beta_j \ge 0 \right\}. \tag{3}$$

The convex polyhedral domains $\mathscr{D}_A$, $\mathscr{D}_B$, with $N = 2^r$, $M = 2^s$ and their vertexes equal to the matrices $A(r)$, $B(s)$, respectively, corresponding to the vertexes i, j of $\mathscr{R}, \mathscr{S}$, are identical with those defined by the hypercubes $\mathscr{R}$, $\mathscr{S}$.

The problem studied in this note is the following: Given the linear uncertain systems described by the model (1)–(3), find a robust control law via linear state feedback $u(t) = -k.x(t)$, such that the closed-loop modes of $(A - B.k)$ lie in the sector γ defined in the Fig. 1, for all the values of the uncertain parameters $r \in \mathscr{R}$, $s \in \mathscr{S}$. For the sake of conciseness, this problem will be named the γ-stabilizability problem and our intent is to find γ-stabilizability conditions for linear uncertain systems and the associated γ-stabilizing state feedback gains.

III. Quadratic Stabilizability and the Augmented System

As said in the introduction, our framework is the quadratic stabilizability concept which is summarized in the following definition.

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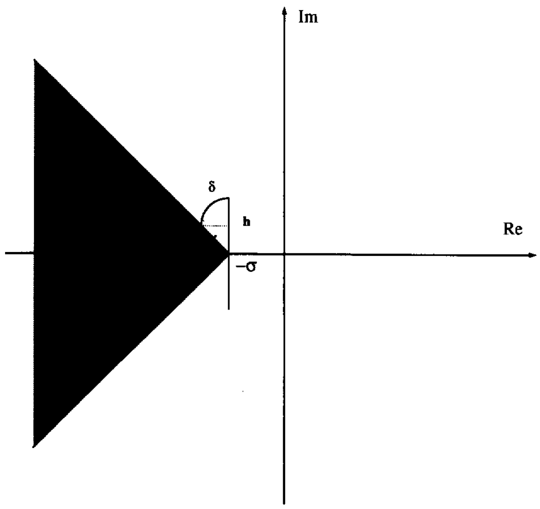

Fig. 1. Sector of root clustering.

Definition 1 [11]: The system (1) is quadratically stabilizable via linear control if there exist a state feedback control $u(t) = -k.x(t)$ and a Lyapunov function $v(x) = x'.P.x$, where P is a symmetric positive-definite matrix, such that (over the uncertainty domains $\mathscr{R}, \mathscr{S}$):

$$x'.[A(r)'.P + P.A(r)].x - 2.x'.P.B(s).k.x < 0 \quad (4)$$

$\forall x \neq 0 \in \Re^n$, and $\forall t \in [0, \infty[$. ◇

Now, let us define the "augmented" system whose properties will allow us to derive the main results. We call "augmented" system, the linear system described by the state-space equations

$$\dot{X}(t) = A_\delta.X(t) + B_\delta.U(t) \quad (5)$$

where

- $A_\delta = \Theta(\delta) \otimes (A + \sigma.I_n)$,
- $B_\delta = \Theta(\delta) \otimes B$,

and $\otimes$ stands for the Kronecker product.

- $P \otimes Q$: Kronecker product of two matrices $P = [p_{ij}] \in \Re^{m.n}$, $Q = [q_{ij}] \in \Re^{p.q}$, defined as follows:

$$P \otimes Q = \begin{pmatrix} p_{11}Q & \cdots & p_{1n}Q \\ \vdots & & \vdots \\ p_{m1}Q & \cdots & p_{mn}Q \end{pmatrix} \in \Re^{mp.nq}$$

- $\Theta(\delta) = \begin{bmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{bmatrix}$,
- $\sigma > 0$.

The order of the system (5) is $2.n$ since, with the definition of the Kronecker product, [4], one notices that: $A_\delta \in \Re^{2.n.2.n}$, $B_\delta \in \Re^{2.n.2.m}$, $X(t) \in \Re^{2.n}$, $U(t) \in \Re^{2.m}$.

Obviously, an uncertain augmented system is derived, taking $A \in \mathscr{D}_A$, $B \in \mathscr{D}_B$, in association with the linear uncertain system (1)

$$\dot{X}(t) = A_\delta(r).X(t) + B_\delta(s).U(t) \quad (6)$$

and

$$\mathscr{D}_{A_\delta} = \left\{ A_\delta \in \Re^{2n.2n}: A_\delta = \sum_{i=1}^{N} \xi_i.A_{i\delta}, \quad \sum_{i=1}^{N} \xi_i = 1 \xi_i \geq 0 \right\}$$

$$\mathscr{D}_{B_\delta} = \left\{ B_\delta \in \Re^{2n.2m}: B_\delta = \sum_{j=1}^{M} \beta_j.B_{j\delta}, \quad \sum_{j=1}^{M} \beta_j = 1 \beta_j \geq 0 \right\}.$$

We show later that the quadratic stabilizability of (6) is closely related to the γ-stabilizability of (1). As for the study of quadratic stabilizability of system (6), we recall a necessary and sufficient condition, whose proof and development can be found in [3].

Theorem 1: The system (6) is quadratically stabilizable via the linear control $U(t) = -K.X(t)$, $K \in \Re^{2m.2n}$ if and only if there exist a positive-definite symmetric matrix $W \in \Re^{2n.2n}$ and a matrix $R \in \Re^{2m.2n}$ such that

$$W.A'_{i\delta} + A_{i\delta}.W < B_{j\delta}.R + R'.B'_{j\delta} \quad (7)$$

$\forall i = 1 \cdots N$, $j = 1 \cdots M$, the robust feedback and a Lyapunov function are given by

$$K = R.W^{-1}, \qquad v(x) = x'.W^{-1}.x. \qquad ◇$$

This theorem is essential for the development of the next results. Indeed, it establishes a condition, whose mathematical properties are very attractive [(7) is convex with respect to the unknown matrices W and R], for the existence of a stabilizing feedback gain for (6).

IV. The Main Results

The first goal is to investigate the links between the quadratic stabilizability of (6) and the γ-stabilizability of (1). The relations between the first problem and the one of the search for a quadratically stabilizing gain for (6) is stated via a necessary and sufficient condition in the following lemma.

Lemma 1: There exists a gain $k \in \Re^{m.n}$, quadratically γ-stabilizing for the system (1) via the state feedback control $u(t) = -k.x(t)$ if and only if there exists a gain $K \in \Re^{2.m.2.n}$, of the form: $K = I_{22} \otimes k$ quadratically stabilizing for the system (6) via the linear control $U(t) = -K.X(t)$. ◇

Proof: This lemma is easily proved using the properties of the Kronecker product [4] and using the fact that stability of the matrix $\Theta(\delta) \otimes A$ is equivalent to the γ-stability of the matrix A [5]. ◇

Thanks to this lemma and to Theorem 1, we are now able to state a necessary and sufficient condition for the quadratic γ-stabilizability of (1).

Lemma 2: The system (1) is quadratically γ-stabilizable via the linear control: $u(t) = -k.x(t)$ iff there exist a positive-definite symmetric matrix $W \in \Re^{2n.2n}$ and a matrix $R \in \Re^{2m.2n}$ such that

$$W.A'_{i\delta} + A_{i\delta}.W < B_{j\delta}.R + R'.B'_{j\delta} \quad (8)$$

for all $i = 1 \cdots N$, $j = 1 \cdots M$, and such that the gain matrix $K = R.W^{-1}$ be of the form: $K = I_{22} \otimes k$, $k \in \Re^{m.n}$. ◇

This lemma clearly shows that the search for the gain K has to be restricted to the class of gains constrained to be block diagonal with same blocks. This structural constraint on the gain K leads to a nonlinear condition with respect to the pair (W, R), which is not easy to handle. A way to overcome this main difficulty, is to convert the structural constraints for the $R.W^{-1}$ product, into structural constraints on the individual W and R matrices. This operation is obviously paid by a loss in the class of the (W, R) matrices. In other words, the necessary and sufficient condition is converted into the sufficient one, given in Theorem 2.

Theorem 2: If there exist a positive definite symmetric matrix $w \in \Re^{n.n}$, a matrix $r \in \Re^{m.n}$ such that

$$H_{ij}(W, R) = W.A'_{i\delta} + A_{i\delta}.W - B_{j\delta}.R - R'.B'_{j\delta} < 0 \quad (9)$$

for all $i = 1 \cdots N$, $j = 1 \cdots M$, and where $W = I_{22} \otimes w$, $R = I_{22} \otimes r$, then the system is quadratically γ-stabilizable via the linear control $u(t) = -k.x(t)$, k given by: $k = r.w^{-1}$.

Obviously, the set of the pair (W, R) which verifies the condition of Lemma 2 is larger than the one defined by the block-diagonal pair (W, R). In this modification, is preserved the linear dependence of the relation (8) with respect to the W and R matrices. This is the fundamental point to be able to define a mathematical programming oriented procedure to determine these unknown matrices.

To our knowledge, there is no analytical way to evaluate the degree of conservativity behind the sufficiency of Theorem 2. The only practical way has been to perform a number of numerical runs. Some of them borrowed from the literature, let us think that the proposed approach is not too conservative. Let $\mathscr{W}$, and $\mathscr{R}$ be the matrix sets:

$$\mathscr{W} = \{W \in \Re^{2n.2n} | W = I_{22} \otimes w, w \in \Re^{n.n}\}$$

$$\mathscr{R} = \{R \in \Re^{2m.2n} | R = I_{22} \otimes r, r \in \Re^{m.n}\}.$$

Associated Optimization Problem: As done in [2], [3], [6], an auxiliary optimization problem is defined so as it enables to check whether the conditions of Theorem 2 define an empty set or not. We propose the following convex program:

A.O.P:

$$\min f(W, R) = \min \rho_1 + \rho_2$$

under

$$\begin{cases} w_{pp} \le \rho_1 \\ -\rho_2 \le r_{qp} \le \rho_2 \end{cases} \quad (10)$$

$$\begin{cases} W \ge \epsilon_1.I_{2n.2n} \\ H_{ij}(W, R) \le -\epsilon_2.I_{2n.2n} \\ W \in \mathscr{W}, R \in \mathscr{R} \\ \forall i = 1 \cdots N, \forall j = 1 \cdots M. \end{cases} \quad (11)$$

The choice of the optimization problem has been done in such a way that good mathematical properties such as convexity are present which, then enables the convergence of the corresponding numerical algorithm. The cost function f and the inequality constraints (10) are classical linear programming formulation and prevent from getting very large norm optimal solutions.

There are several ways in order to deal with such global optimization problems. The constraints (11) define a conic convex set for the W, R matrices, however, their analytical expressions in terms of the unknowns (the matrices entries) is highly nonlinear. The cutting plane technique has been chosen, its advantage is to iteratively approximate the nonlinear constraints by a sequence of linear ones. The algorithm forms a series of improving approximating linear programs whose solutions converge to the solution of the original problem. We start from an initial polytope defined by (10), containing the convex domain, and then, elaborate successive cuts approximating this convex domain. The separating hyperplanes generating the new linear convex constraint are calculated relatively to the most violated constraint (11).

Numerical Procedure:

Step 4.1)

Initialization:

$$l = 0, \quad W_l = W_1 = \boldsymbol{I_{2n.2n}} \quad \text{and} \quad R_l = R_1 = \boldsymbol{0_{2m.2n}}.$$

Step 4.2)

Calculate:

$$\lambda_W = \lambda_{\min}(W_l)$$

and

$$\lambda_{H_{ij^*}} = \max_{i,j} \lambda_{\max}(H_{ij}(W_l, R_l)).$$

If $\lambda_W \ge \epsilon_1$ and $\lambda_{H_{ij^*}} \le -\epsilon_2$ STOP:

The pair (W_l, R_l) satisfies the condition and is a solution.

Else: $l = l + 1$ and calculate the normalized eigenvectors ν_W and $\nu_{H_{ij^*}}$ associated respectively with λ_W and $\lambda_{H_{ij^*}}$.

Calculate the constraint (linear):

$$C_l(W, R) = \begin{cases} \nu'_W.W_l.\nu_W \ge \epsilon_1 & \text{if } (\epsilon_1 - \lambda_W) > (\lambda_{H_{ij^*}} + \epsilon_2) \\ \nu'_{H_{ij^*}}.H_{ij^*}.\nu_{H_{ij^*}} \le -\epsilon_2 & \text{if } (\epsilon_1 - \lambda_W) \le (\lambda_{H_{ij^*}} + \epsilon_2). \end{cases}$$

Step 4.3)

Solve:

$$\min \rho_1 + \rho_2$$

under

$$\begin{cases} \epsilon_1 \le w_{pp} \le \rho_1 \\ -\rho_2 \le r_{qp} \le \rho_2 \\ C_k(W, R) \quad k = 1, \cdots, z \end{cases}$$

with $p = 1, \cdots, 2n$ $q = 1, \cdots, 2m$.

If there is no solution STOP:

no pair (W_l, R_l) exists which satisfy Theorem 2.

Else if (W_l, R_l) solutions, return to Step 4.2).

The convergence and the rate of convergence of such an algorithm have been studied in [6].

V. Numerical Example

The example is borrowed from a note of W. E. Schmitendorf dealing with an uncertain model of the dynamics of a helicopter in a vertical plane (see [16] for more details). The uncertain dynamical model is the following:

$$\dot{x}(t) = \begin{pmatrix} -0.00366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 + a_{32} & -0.707 & 1.3229 + a_{34} \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 0.4422 & 0.1761 \\ 3.0447 + b_{21} & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{pmatrix} . u(t) \quad (12)$$

with

$$|a_{32}| \le 0.2192, \quad |a_{34}| \le 1.2031, \quad |b_{21}| \le 2.0673.$$

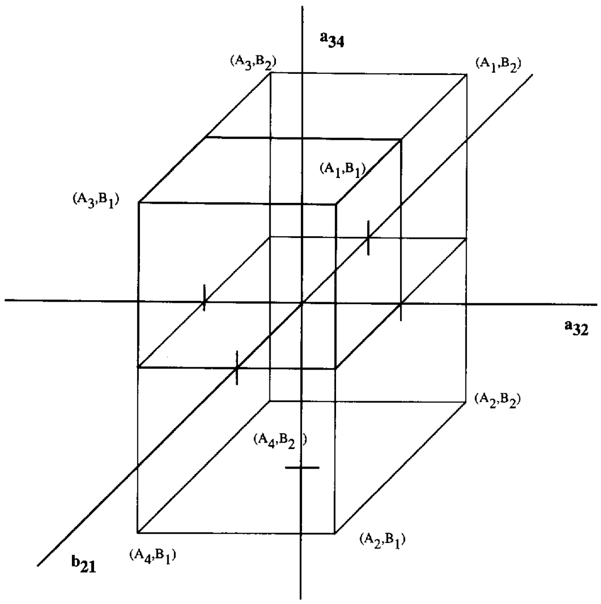

Fig. 2. Uncertainty hypercube.

Thus, the convex polyhedral domains $\mathscr{D}_A$, $\mathscr{D}_B$ with, respectively, $N = 4$, $M = 2$ vertexes, are defined by their extremal matrices

$$A1 = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.5047 & -0.707 & 2.526 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A2 = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.5047 & -0.707 & 0.12 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A3 = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.0663 & -0.707 & 2.526 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A4 = \begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.0663 & -0.707 & 0.12 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B1 = \begin{pmatrix} 0.4422 & 0.1761 \\ 5.112 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{pmatrix} \qquad B2 = \begin{pmatrix} 0.4422 & 0.1761 \\ 0.9774 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{pmatrix}$$

We are interested in the issue of the conservativeness of the sufficient condition of Theorem 2. For the fixed uncertainty hypercube defined for the model (12), (Fig. 2) and for a fixed null relative stability degree, we look for the largest angle of the wedge allowed. For $\delta = 82^0$, we get after 136 iterations

$$W = \begin{pmatrix} 308.52 & -9.85 & -1.01 & 3.04 \\ -9.85 & 308.52 & -54.60 & 7.97 \\ -1.01 & -54.60 & 40.57 & -5.12 \\ 3.04 & 7.97 & -5.12 & 1.68 \end{pmatrix}$$

$$R = \begin{pmatrix} -0.57 & -52.43 & -27.33 & 1.52 \\ 7.41 & -282.60 & 142.89 & -4.60 \end{pmatrix}$$

$$K = R.W^{-1} = \begin{pmatrix} -0.004 & -0.37 & -1.36 & -1.46 \\ -0.11 & -0.45 & 4.64 & 13.67 \end{pmatrix}.$$

In the context of uncertain linear system control, the problem of performance, in terms of transient response, has not been very often investigated and it still lacks of a strong and effective design procedure. Although the present synthesis procedure is only sufficient, its relative simplicity and effectiveness on several examples borrowed from the literature, let us think that it provides a partial but valuable answer.

Our approach allows to find a unique Lyapunov function for all the uncertainty domain and simultaneously gives the single γ-stabilizing feedback gain. Of course, there are still numerous questions under investigations. One of them consists in the extension of this approach to different subregions of the complex plane using the works of [13] and particularly the case of discrete-time systems has to be investigated. Another point to be carefully studied, is the associated optimization problem whose form governs the characteristics of the gain [6], and therefore the performances. Specially, it remains to be seen how to relax the structural constraint on R and W, in order to enlarge the search domain.

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Feedback Decoupling of Structured Systems

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***Abstract*—In this note we consider the feedback decoupling of structured linear systems. Gathering results on decoupling and the graph characterization of the structure at infinity, we give a necessary and sufficient decoupling condition for structured systems.**

This condition has a nice graphical interpretation in terms of shortest input–output paths.

I. Introduction

Due to their appealing physical interpretation structured systems have received a great deal of attention in the past ten years. A structured state space representation is a triplet (A,B,C) where the entries of (A,B,C) are either null or mutually independent parameters.

For such structured systems one can study structural properties i.e., properties which are true for almost all values of the parameters. A useful tool for this purpose is the graph representation of such systems. The structural controllability has been studied for example in [1], [2].

In this note, following [10], we give a new necessary and sufficient feedback decoupling condition which reduces to the comparison of two structural integers.

When applied to structured systems, this condition has a nice graphical interpretation in terms of shortest input–output paths in the associated graph. As a corollary, we show that this result is equivalent to the decoupling condition established previously in [6].

The note is organized as follows: in Section II and III the basic definitions concerning structured systems and associated graphs, infinite structure, and its graphical interpretation for structured systems are recalled. The new feedback decoupling condition is given in Section IV. Section V is devoted to the very simple graphical interpretation of this condition for structured systems.

II. Structured Systems and Associated Graphs

A structured matrix is a matrix whose entries are either fixed zeros or independent parameters [1], [6]. Denote by Λ the vector composed of the k nonnull parameters of a structured matrix, then $\Lambda \in \mathrm{V}$ where V is an open subset of $\Re^k$.

The system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$u \in \Re^m, x \in \Re^n, y \in \Re^p$$

is a *structured system* if the matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \text{ is structured.}$$

A structured system with parameter set Λ will be denoted by Σ_Λ.

For Σ_Λ a property which is true for any $\Lambda \in \mathrm{V}$ except on the intersection of V with a proper algebraic variety of $\Re^k$ is called a *structural property*.

Recall that a proper algebraic variety in $\Re^k$ is defined as the set $\lambda = (\lambda_1, \cdots, \lambda_k)^T$ of common zeros of a finite number of nontrivial polynomials in λ_i with real coefficients.

Then a structural property will be true for almost all systems with the required structure. Furthermore, if for a given system the property is not true it will be true for some arbitrarily close system.

Some properties of structural systems have been studied in the past years. For example nice characterizations of structural controllability have been given [1], [2].

We will associate to the above defined structured system Σ_Λ a graph $G(\Sigma_\Lambda) = \{Z, W\}$ where the vertex set is

$$Z = U \cup X \cup Y$$

where

$$U = \{u_1, \cdots, u_m\}$$
$$X = \{x_1, \cdots, x_n\}$$
$$Y = \{y_1, \cdots, y_p\}$$

and the edge set is defined as follows:

$$W = \{(u_i, x_j) | b_{ji} \neq 0\} \cup \{(x_i, x_j) | a_{ji} \neq 0\} \cup \{(x_i, y_j) | c_{ji} \neq 0\}$$

where b_{ji} (respectively a_{ji}, c_{ji}) denotes the element (j, i) of the matrix B (respectively A, C). A path which joins inputs to outputs, through state vertexes, is called an input–output path. For an illustrative example see [3].

III. Graph Characterization of the Infinite Structure

In this section, we recall briefly the graph interpretation of the infinite structure which was introduced in [3].

Let $T(s)$ be a $(p \times m)$ proper rational matrix. $T(s)$ can be factorized as follows:

$$T(s) = B_1(s) \begin{bmatrix} s^{-n_1} & & & & \\ & \ddots & & & \\ & & s^{-n_r} & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} B_2(s)$$
$$= B_1(s)\Lambda(s)B_2(s)$$

where

$$\Lambda(s) = \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta(s) = \operatorname{diag}(s^{-n_1}, \cdots, s^{-n_r}).$$

$B_1(s)$ and $B_2(s)$ are bicausal matrices (proper, invertible and of proper inverse) characterized by:

$$\det\left(\lim_{s \to \infty} B_i(s)\right) \text{ is a nonnul constant} \qquad i = 1, 2$$

$$r = \operatorname{rank} T(s).$$

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