

# Perspectives on Johnson Noise

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## 1 Background

In electrical circuits, the noise due thermal fluctuations in a resistor is called Johnson noise. Some important applications are discussed in [section 4](#).

The question arises, what happens if there are two resistors in parallel? How much noise does the combination produce? That question is particularly interesting because there are two different ways of answering it. It is amusing to do it both ways and think about why two calculations that look so different produce the same result.

The Johnson noise is well described by the famous Nyquist formula:

$$\langle V^2 \rangle = 4 k T B R \quad (1)$$

where  $V$  is the voltage,  $kT$  is the temperature (in energy units),  $B$  is the frequency (in circular measure, not radians), and  $R$  is the resistance. The brackets  $\langle \dots \rangle$  denote an ensemble average.

Algebraically, [equation 1](#) could hardly be simpler, although conceptually you have to think about it for a while before you really understand it. It is a direct consequence of the laws of motion and basic thermodynamics, i.e. the idea that there is  $\frac{1}{2}kT$  of thermal energy per quadratic degree of freedom, combined with a clever way of enumerating the degrees of freedom in the resistor.

Johnson and Nyquist were colleagues; they published back-to-back papers in The Physical Review in 1928 ([reference 1](#) and [reference 2](#)).

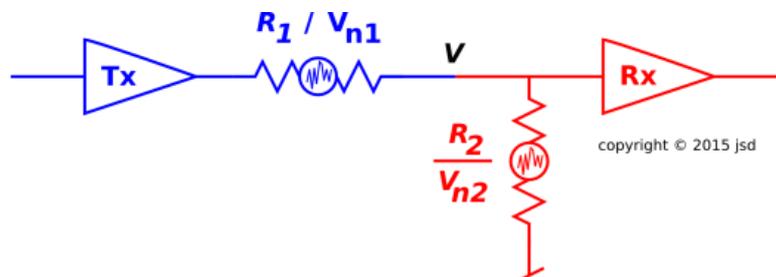
We will not rederive the Nyquist formula here. Instead, we take it as a starting point and investigate some of the consequences. In [section 2](#) we explore what happens when there are two resistors are in parallel. Then in [section 3](#), we explore what happens if there is a capacitor in the circuit, resulting in a frequency-dependent noise voltage.

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## 2 Noise in a Resistive Voltage Divider

Consider the circuit shown in [figure 1](#). Circuits like this show up all the time. We can consider  $R_1$  to be the output impedance of the line driver (Tx), while  $R_2$  is the input impedance of the receiver (Rx). The Tx triangle itself is an ideal zero-impedance voltage source; whatever impedance is associated with the driver is included in  $R_1$ . Similarly, the Rx triangle itself is an ideal infinite-impedance voltage measuring device; whatever input impedance is associated with the receiver is included in  $R_2$ .



[Figure 1](#): Resistor/Resistor Noise Divider

Suppose that Tx is putting out zero volts, so that the only voltages in the system are the noise voltages. We are particularly interested in the voltage  $V$  at the point where the two subsystems meet. Note that this voltage appears directly at the input to the Rx triangle.

The Johnson noise in  $R_1$  is a distribution over voltages. In accordance with [equation 1](#), the distribution is characterized by the mean square voltage, namely:

$$\langle V_{n1}^2 \rangle = 4 k T B R_1 \quad (2)$$

A similar formula applies to  $R_2$ .

We start by considering the limit where  $R_1$  is very large compared to  $R_2$ . That means the noise voltage associated with  $R_1$  is very large. However, that voltage gets reduced by a factor of  $g$ , where  $g$  is less than 1 and represents the “gain” of the voltage divider formed by  $R_1$  and  $R_2$ , namely:

$$g := \frac{R_2}{R_1 + R_2} \quad (3)$$

Reducing the voltage by a factor of  $g$  reduces the mean *square* voltage by a factor of  $g$  squared. So all in all, the contribution to  $V$  from  $R_1$  is:

$$\langle V_1^2 \rangle = 4 k T B \frac{R_1 R_2^2}{(R_1 + R_2)^2} \quad (4)$$

In the limit where  $R_1$  becomes large, its contribution becomes negligible. The only noise we see comes from  $R_2$ . Qualitatively this makes sense; an infinite resistor is equivalent to an open circuit. It might as well not be there.

The two resistors play completely symmetrical roles. The corresponding formula for the contribution from  $R_2$  is:

$$\langle V_2^2 \rangle = 4 k T B \frac{R_2 R_1^2}{(R_1 + R_2)^2} \quad (5)$$

Now we need to add these two contributions. If we knew the voltages, we would just add them, but we don't. All we have is a statistical distribution over voltages. Taking the square root of [equation 4](#) or [equation 5](#) will not tell you “the” voltage; instead it tells you the width of the distribution over voltages.

Adding uncorrelated random voltages means the widths will add *in quadrature*. So, the distribution on  $V$  is characterized by a mean square voltage:

$$\langle V^2 \rangle = 4 k T B \frac{R_1 R_2^2 + R_2 R_1^2}{(R_1 + R_2)^2} \quad (6)$$

There is another way of figuring out the noise voltage. We could just recognize that the effective resistance is the parallel combination of  $R_1$  and  $R_2$ :

$$\begin{aligned} R_{\text{eff}} &= R_1 \parallel R_2 \\ &= \frac{R_1 R_2}{R_1 + R_2} \end{aligned} \quad (7)$$

We use the Nyquist formula directly to calculate the noise in this resistor:

$$\begin{aligned} \langle V^2 \rangle &= 4 k T B R_{\text{eff}} \\ &= 4 k T B \frac{R_1 R_2}{R_1 + R_2} \end{aligned} \quad (8)$$

Using a little algebra, you can confirm that [equation 6](#) and [equation 8](#) are equivalent.

**Discussion:** The most important part of the calculation is realizing that it can be done in more than one way!

It is amusing that two wildly different methods of calculation arrive at the same answer, namely [equation 6](#) which agrees with [equation 8](#). This didn't happen by accident. It is connected to some fundamental principles of thermodynamics.

For one thing, it is important to look closely at the starting point. In [figure 1](#), we assumed that the two resistors were in thermal equilibrium with each other. Thermodynamics tells us that subject to mild provisos, equilibrium is isothermal. That is to say, two objects in thermal equilibrium are at the same temperature. Even though the resistance-values are different, the temperatures are the same (at equilibrium). The main proviso is that each object must be big enough and complicated enough to serve as a “heat bath” for the other. For reasons explained in [reference 2](#), if you include the resistor and the heat sink to which it is bolted, it makes an extraordinarily good heat bath.

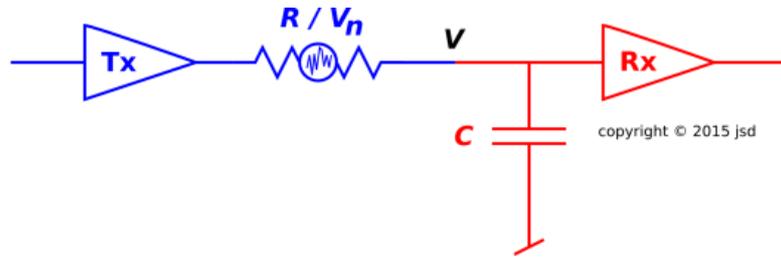
Furthermore, we have the idea of equivalent circuit. If  $R_1$  in parallel with  $R_2$  is equivalent to  $R_{\text{eff}}$  – equivalent for the purposes of Ohm's law – the theoretical model in [reference 2](#) says it should be equivalent for the purposes of thermodynamics also. It should have the same density of states, and therefore the same fluctuations. If [equation 6](#) did not match [equation 8](#), it would tell us that the model was wrong.

This is how a good bit of theoretical physics gets done: You find something that can be calculated two different ways, and compare the results. Most of the time the results agree, but if they disagree you'll learn something by figuring out why.

Another example of calculating something two different ways leads to [equation 12](#) in [section 3](#). Among other things, that tells us that the factor of 4 in [equation 1](#) is not a mistake; it is the only possible value that is consistent with fundamental notions of thermodynamics.

### 3 Noise in a RC Voltage Divider

We now turn to the resistor/capacitor voltage divider in [figure 2](#). This is different from the situation we considered in [section 2](#), because the capacitor divides the noise voltage coming from  $R$  but does not add any noise of its own.



[Figure 2](#): Resistor/Capacitor Noise Divider

As always, the noise in  $R$  is a distribution over voltages, as given by [equation 2](#). This is a linear circuit, so we can analyze it frequency by frequency. At any given frequency  $f$ , the voltage that appears at node  $V$  will be less  $V_n$  by a factor of

$$g = \frac{1}{\sqrt{1 + (f/f_0)^2}} \quad (9)$$

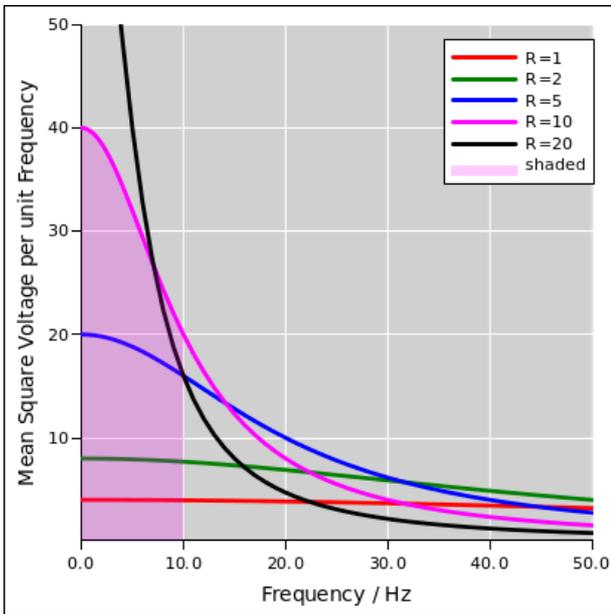
where

$$f_0 := \frac{1}{2\pi R C}$$

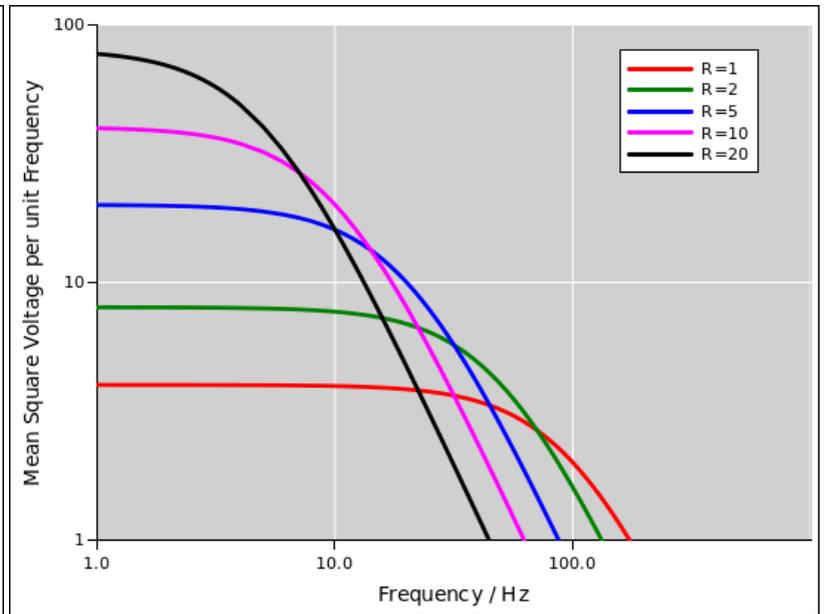
Putting the ingredients together, we find that the noise at node  $V$  in any band  $B$  is:

$$\langle V^2 \rangle = 4 k T R \int_B \frac{1}{1 + (2\pi R C_1 f)^2} df \quad (10)$$

The noise density (i.e. the integrand in [equation 10](#)) is shown in [figure 3](#) for a fixed capacitor and various different resistor values. [Figure 4](#) shows the same data on log/log axes.



[Figure 3](#): Noise per Unit Frequency



[Figure 4](#): Noise per Unit Frequency, Log/Log

The integral in [equation 10](#) can be expressed in terms of the arctangent. Integrating it from zero to the corner frequency  $f_0$  gives  $f_0 \pi/4$ . Integrating it from zero to infinity gives  $f_0 \pi/2$ , which is exactly twice as much. The shaded region in [figure 3](#) is half of the total area under the magenta curve.

$$\begin{aligned} \langle V^2 \rangle &= 4 k T R f_0 \pi/2 \\ &= 4 k T R \frac{\pi/2}{2\pi R C} \\ &= \frac{kT}{C} \end{aligned} \quad (11)$$

where the last line is, remarkably enough, independent of the resistance. We have an  $RC$  circuit, and the physical origin of the noise is in the resistor, but the magnitude of the noise is independent of the resistance, for any circuit of this kind.

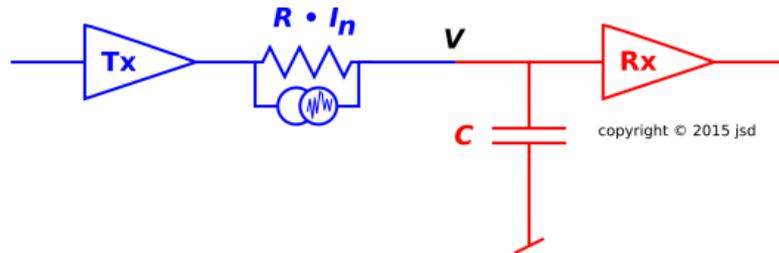
In other words, the area under the curve is the same for all curves in [figure 3](#). For our example circuit, the area is  $200\pi$  in the appropriate units. That's because we have chosen  $kT=1$  and chosen the capacitor value to be  $.01/2\pi$ . Those values were chosen so as to make the corner frequencies come out to be round numbers when expressed in hertz.

In particular, since the resistance does not appear in the last line of [equation 11](#), we are free to consider the case where  $R \rightarrow \infty$ , which is a capacitor all by itself. In this case the noise density curve (as in [figure 3](#)) becomes a delta function; there is a distribution of voltages at zero frequency and nowhere else. That is to say, the voltage on the capacitor is not fluctuating, but if we take an ensemble of capacitors, there will be a spread in voltage. The average energy on the capacitor is

$$\langle \frac{1}{2} C V^2 \rangle = \frac{1}{2} kT \quad (12)$$

which agrees with basic notions of thermodynamics.

Note that it is always possible to replace a noise voltage in series with a noise current in parallel. Compare [figure 5](#) with [figure 2](#).



[Figure 5](#): Norton Equivalent

The mean square noise current is *inversely* proportional to resistance. Compare [equation 13](#) with [equation 2](#).

$$\langle I_n^2 \rangle = 4 kT B / R \quad (13)$$

This is consistent with Ohm's law: A current proportional to  $1/\sqrt{R}$ , multiplied by  $R$ , gives a voltage proportional to  $\sqrt{R}$ .

This gives us another way to understand the large- $R$  behavior of the circuit in [figure 2](#). The capacitor is fed by a very small noise current. The capacitor integrates this current over a very long time. If you wait long enough, the charge on the capacitor will random-walk a very long way ... but you do have to wait.

## 4 Applications, Implications, and Ramifications

Johnson noise is relevant in a tremendous variety of circuits. This is an example of how thermodynamics applies in real-world situations.

- If you are trying to build a low-level high-bandwidth amplifier, you want to minimize the noise. Thermal fluctuations establish an inescapable noise floor ... assuming the amplifier treats all phases equally. (You can do better using a phase-sensitive amplifier, as discussed in [reference 3](#).)
- If you are trying to build an electronic random number generator, you don't want to minimize the noise; you want to cultivate it and collect it. The results of [section 3](#) are directly relevant in this case. If the input capacitance and input resistance give you a large  $RC$  time, large compared to the sampling interval of the digitizer, the noise will not be white. By adding a shunt resistor, you can make the noise whiter, but it will be smaller in magnitude.

## 5 References

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