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Design Formulas for Biquad Active Filters Using Three Operational Amplifiers

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Abstract—**A circuit configuration and its design formulas are presented for the realization of all the useful forms of a biquadratic voltage transfer function. The circuit employs three single-ended operational amplifiers, two capacitors, and at most eight resistors. With an additional resistor, it can realize any biquadratic voltage transfer function.**

INTRODUCTION

It has been known that some biquadratic voltage transfer functions can be realized with three single-ended operational amplifiers, two capacitors, and at most eight resistors [1], [2]. However, the design formulas for computing the element values from the coefficients of the transfer function have not been fully published before. The purpose of this letter is to show a general circuit configuration and present the corresponding design formulas.

CIRCUIT CONFIGURATION AND DESIGN FORMULAS

Let the general biquadratic voltage transfer function be given by

$$\frac{V_{\text{out}}}{V_{\text{in}}}(s) = \frac{ms^2 + cs + d}{s^2 + as + b} \tag{1}$$

where it is tacitly assumed that the poles are complex and the circuit is stable, i.e., $a>0$ and $b>a^2/4$. The circuit[1] in Fig. 1 can be used to realize (with a possible change of sign) all of the following useful forms of (1):

low-pass	$m=c=0$
bandpass	$m=d=0$
high-pass	$c=d=0$
band-elimination	$m>0,\ c=0,\ d>0$
all-pass	$c=-ma,\ d=mb.$

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[1] The circuit in Fig. 1 is identical to the circuit in Fig. 10 of [1]; however, design formulas were given only for the band-elimination case in [1].

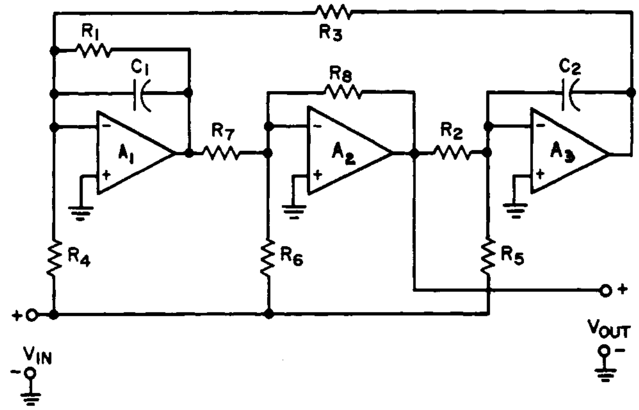

Fig. 1. Multiple-input biquad circuit diagram.

complex zero section where the zeros are located to the right of the poles in the complex frequency plane:

$$m>0,\quad c\neq 0,\quad d>0,\quad \text{and}\quad (ma-c)\geq 0.$$

The transfer function of the circuit in Fig. 1 can easily be shown to be

$$\frac{V_{\text{out}}}{V_{\text{in}}}(s) = -\frac{\dfrac{R_8}{R_6}s^2 + \dfrac{1}{R_1C_1}\left[\dfrac{R_8}{R_6} - \dfrac{R_1R_8}{R_4R_7}\right]s + \dfrac{R_8}{R_3R_5R_7C_1C_2}}{s^2 + \dfrac{1}{R_1C_1}s + \dfrac{1}{R_2R_3C_1C_2}\cdot\dfrac{R_8}{R_7}}. \tag{2}$$

By matching the coefficients of (1) and (2), a set of (positive) element values is obtained as follows:

$$\begin{aligned} R_1 &= \frac{1}{aC_1} & R_2 &= \frac{k_1}{\sqrt{b}\,C_2} \\ R_3 &= \frac{1}{k_1k_2}\cdot\frac{1}{\sqrt{b}\,C_1} & R_4 &= \frac{1}{k_2(ma-c)C_1} \\ R_5 &= \frac{k_1\sqrt{b}}{dC_2} & R_6 &= \frac{1}{m}R_8 \\ R_7 &= k_2R_8 \end{aligned} \tag{3}$$

where C_1, C_2, R_8, k_1, and k_2 are the free parameters. Note that depending on the numerator coefficients, some of the "feed-in" resistors R_4, R_5, or R_6 may become infinite.

The values of C_1, C_2, and R_8 control impedance levels and are chosen to yield convenient element values.

The choice of element values according to (3) may be shown to result in the following transfer functions:

$$\frac{V_{\text{out}}}{V_{\text{in}}}(s) = -\frac{ms^2 + cs + d}{s^2 + as + b}$$

$$\frac{V_1}{V_{\text{in}}}(s) = -k_2\frac{(ma-c)s + (mb-d)}{s^2 + as + b}$$

$$\frac{V_3}{V_{\text{in}}}(s) = -\frac{1}{k_1}\cdot\frac{\dfrac{d-mb}{\sqrt{b}}s + \dfrac{ad-bc}{\sqrt{b}}}{s^2 + as + b}.$$

Thus the parameters k_1 and k_2 may be chosen to establish the maximum voltage levels at the other two amplifiers. Alternatively, for minimum sensitivity design ($R_3C_1 = R_2C_2$) it may be desirable to set $k_1^2k_2 = 1$.

OTHER CASES

The circuit in Fig. 1, together with its design formulas (3), can be used to realize, in addition to the foregoing useful forms, any biquadratic voltage transfer function with the exception of the following three cases.

Case 1: Complex zeros or two real left-half plane zeros where the "center of gravity" of the zeros lies to the left of the poles in the complex frequency plane, i.e., $(ma-c)<0$ and $d\geq 0$.

Case 2: The numerator contains only one real zero which is in the left-half plane, i.e., $m=0$, c and d are of the same sign.

Case 3: One real left-half plane and one real right-half plane zeros, i.e., m and d are of opposite signs.

To realize these cases, the following circuit modifications may be made.

Case 1: Add a positive feedback resistor between the output of A_2 and the input of A_1.

Case 2: Use circuit of Fig. 1 but take the output from amplifier A_3.

Case 3: Interchange R_8 and C_2, add a positive feedback resistor from A_3 to A_2, and take the output from A_3. Because these cases rarely arise in practical filter design, the corresponding design formulas are not given here.

Conclusion

Complete design formulas are given for a three-operational amplifier biquad realization for all of the useful biquadratic voltage transfer functions. This circuit realizes the transmission zeros by an input feed-forward technique instead of the summation technique described in [2] and [3] where a fourth amplifier is also needed. It also requires one less capacitor than the circuit described in [4]. The low sensitivity and noninteractive tuning properties of [1]–[4] are also preserved.

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Hilbert Transform Relations for Products

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***Abstract*—A general formula for the Hilbert transform of a product of complex-valued functions is developed. Certain simplifications are then exhibited for products often encountered in the context of modulation and signal processing. The approach chosen is one of frequency partitioning; this permits signal definition on complementary sections of the frequency axis and leads to compact and easily manipulated expressions.**

Introduction

The Hilbert transform is a useful analytical tool that has been applied extensively in signal and system theory. While this transform provides a tidy means of relating certain orthogonal time or frequency functions, the actual computation of transform pairs and the reduction of transform expressions is usually a difficult task.

One important problem is that of finding Hilbert transforms of products since most useful applications are replete with instances requiring multiplication of functions. Product relationships came under scrutiny in connection with narrow-band signal representation. Some controversy [1]–[7] surrounded this matter before Bedrosian's product theorem was introduced [4], [7]. Bedrosian's theorem, however, is applicable only in circumstances that are often overly restrictive. The purpose of this letter is to present a more general result and to show the relevance of work by Tricomi [8] and Titchmarsh [9]; also, some special simplifications for expressions common in modulation and signal processing analyses are demonstrated. These results can be used to advantage in a variety of specific situations (for example, [10]).

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The Hilbert transform of $g(t)$ is a linear operator defined as

$$\mathcal{H}[g(t)] = \hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\, d\tau$$

where the integration is taken in the Cauchy principal value sense and where $\mathcal{H}[\cdot]$ and $\hat{}$ are equivalent symbology. If t is identified as a time variable, the Hilbert transform may be viewed as a $-90°$ phase-shift operator; such a viewpoint is often beneficial in understanding the usefulness of this transform. Since constants are lost in Hilbert transformation, we stipulate, in order to provide for unique inverse transformation, that no functions with dc components be allowed. It will be assumed that all indicated Hilbert transforms are being taken in a distributional sense so that we can cater for "power"-type signals (such as sinusoids) which so frequently arise in communication studies, as well as the finite energy signals originally specified in [4].

Bedrosian's theorem for (generally) complex-valued time functions may, for convenience of application, be paraphrased as follows:

$$\mathcal{H}[x(t)y(t)] = x(t)\hat{y}(t) \tag{1}$$

if:

a) denoting f_p as the smallest frequency value along the positive f axis at which the Fourier transform $Y(f)$ is nonzero, it is found that $X(f)$ vanishes below $-f_p$;

and furthermore if:

b) labeling as $-f_n$ the largest frequency value along the negative f axis at which $Y(f)$ is nonzero, we find that $X(f)$ vanishes above f_n.

If $Y(f)$ happens to be nonzero throughout an interval adjoining the origin, we are confronted with the special problem of accommodating the point $f=0$ in any declaration of f_p or $-f_n$. This problem can be overcome (and Bedrosian's theorem successfully applied) by the conceptual artifice of sidestepping the frequency origin; that is, for a continuous spectrum $Y(f)$ which is contiguous to the frequency origin from the positive frequency side, it is necessary to assign f_p an arbitrarily small positive value, say ϵ_p. Similarly, if on the negative frequency axis $Y(f)$ is nonzero on an interval adjacent to the origin, then $-f_n$ is chosen to be the small negative offset $-\epsilon_n$. This means, for instance, that if $Y(f)$ has nonzero continuous spectral content straddling the origin, a delta function at the origin (that is, a constant $x(t)$, which is not allowable due to our initial stipulation) is the only spectrum for $X(f)$ that would, according to this theorem, satisfy (1).

Clearly, Bedrosian's theorem is not generally applicable for spectrally overlapping double-sided baseband signals.

In the case of a single-sided $Y(f)$ that vanishes along the positive (or negative) frequency axis, no f_p (or $-f_n$) is encountered; therefore, there is no constraint on the extent of $X(f)$ for negative (positive) frequency. It is important to note that a) and b) are sufficiency conditions only and that (1) might, in some examples, hold true even if these conditions fail.

One of the simplest cases meeting the qualifications of the theorem is that of a low-pass signal $x(t)$ strictly band-limited inside $(-f_c, f_c)$ combined with a high-pass signal $y(t)$ with no spectral content inside that interval. Another obvious case is that in which both signals are analytic; this case will enable us to extend Bedrosian's theorem. We can form analytic signals from real-valued r and s:

$$x_0(t) = r(t) + j\hat{r}(t)$$
$$y_0(t) = s(t) + j\hat{s}(t).$$

Employing (1) and equating real parts, we are led to an expression that is independent of spectral considerations:

$$\mathcal{H}[r(t)s(t)] = r(t)\hat{s}(t) + \hat{r}(t)s(t) + \mathcal{H}[\hat{r}(t)\hat{s}(t)]. \tag{2}$$

Equation (2) seems to have been derived first (and more rigorously than was done here) by Tricomi [8], although it appears to have seen little use in engineering literature.

We will now obtain a new equation with the form of (2) which is true for complex-valued functions. We first introduce two new arbitrary real-valued functions $v(t)$ and $w(t)$. Then three additional equations like (2) are written out, where rs is replaced first by $-vw$, then