

Table for Fundamentals of Series: Part I: Basic Properties of Series and Products

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1 Binomial Identities

Remark 1.1 *Throughout these tables, we assume, unless specified, that n, j, k, α and r represent non-negative integers. Furthermore, we reserve x , and y for arbitrary real(complex) numbers.*

1.1 Basic Identities

Pascal's Formula

$$\binom{x}{k} = \binom{x+1}{k} - \binom{x}{k-1} \quad (1.1)$$

Committee/Chair Identity

$$(n+1)\binom{r}{n+1} = r\binom{r-1}{n} \quad (1.2)$$

Cancelation Identity

$$\binom{n}{r+a}\binom{r+a}{r} = \binom{n}{r}\binom{n-r}{a} \quad (1.3)$$

$$\binom{n}{r}\binom{n+r}{r} = \binom{n+r}{n-r}\binom{2r}{r} = \binom{n+r}{2r}\binom{2r}{r} = \frac{(n+r)!}{(r!)^2(n-r)!} \quad (1.4)$$

$$\binom{n}{r}\binom{2n}{n} = \binom{n+r}{r}\binom{2n}{n-r} \quad (1.5)$$

-1 Transformation

$$\binom{-x}{r} = (-1)^r \binom{x+r-1}{r} \quad (1.6)$$

$$\binom{n+r}{r} = \binom{n+r}{n} = (-1)^n \binom{-r-1}{n} \quad (1.7)$$

$$\frac{\binom{-x+\alpha-1}{n+r} \binom{x}{\alpha-r-n}}{\binom{x}{\alpha}} = (-1)^{n+r} \binom{\alpha}{n+r} \quad (1.8)$$

$\frac{-1}{2}$ Transformation

$$\binom{\frac{-1}{2}}{n} = (-1)^n \binom{2n}{n} \frac{1}{2^{2n}} \quad (1.9)$$

$\frac{1}{2}$ Transformation

$$\binom{\frac{1}{2}}{n} = (-1)^{n+1} \binom{2n}{n} \frac{1}{2^{2n}(2n-1)} \quad (1.10)$$

$$\binom{\alpha}{n+1} \binom{\alpha-n-1}{\alpha-j} \binom{n+1}{j-k} = \binom{\alpha}{j} \binom{j}{k} \binom{k}{j-n-1} \quad (1.11)$$

1.2 Binomial Identities From the Gamma Function

Identities from $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

$$n!(-1-n)! = \frac{\pi}{\sin(n+1)\pi} \quad (1.12)$$

$$\left(-n - \frac{1}{2}\right)! \left(n - \frac{1}{2}\right)! = (-1)^n \pi \quad (1.13)$$

Identities from Duplication Formula: $\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right)$

$$\left(n - \frac{1}{2}\right)! = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!} \quad (1.14)$$

$$\frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{n!\sqrt{\pi}}{2\left(n + \frac{1}{2}\right)!} \quad (1.15)$$

$$\left(\frac{n}{2}\right)! \left(\frac{n-1}{2}\right)! = \frac{n!\sqrt{\pi}}{2^n} \quad (1.16)$$

$$\binom{n}{\frac{1}{2}} = \frac{2^{2n+1}}{\pi \binom{2n}{n}} \quad (1.17)$$

$$\binom{k}{\frac{1}{2}} \binom{n-k}{\frac{1}{2}} = \frac{2^{2n+2}}{\pi^2 \binom{2k}{k} \binom{2n-2k}{n-k}} \quad (1.18)$$

$$\binom{n}{\frac{n}{2}} = \frac{2^{2n} \left(\frac{n-1}{2}\right)!}{n!\pi} = \frac{2^{2n}}{\pi \binom{n-1}{\frac{n-1}{2}}} \quad (1.19)$$

1.3 Limit Formulas

$$\lim_{n \rightarrow \infty} \binom{2n}{n}^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n)!}{(n!)^2}} = 4 \quad (1.20)$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n]{n!}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e \quad (1.21)$$

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{n!} = 1 \quad (1.22)$$

2 Series: The Basic Properties

2.1 Indices Properties

Remark 2.1 In this chapter, we assume a is a nonnegative integer. We also let $[x]$ denote the floor of x , i.e. the greatest integer less than or equal to x .

$$\sum_{k=a}^n f(k) = \sum_{k=0}^n f(k) - \sum_{k=0}^{a-1} f(k) \quad (2.1)$$

$$\sum_{j=k_0+1}^{k_m} f(j) = \sum_{i=0}^{m-1} \sum_{j=k_i+1}^{k_{i+1}} f(j), \quad 1 \leq m \leq \infty \quad (2.2)$$

$$2 \sum_{k=\lceil \frac{a+1}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} f(2k) = \sum_{k=a}^n f(k) + \sum_{k=a}^n (-1)^k f(k), \quad n \geq a+1 \quad (2.3)$$

$$2 \sum_{k=\lceil \frac{a+2}{2} \rceil}^{\lfloor \frac{n+1}{2} \rfloor} f(2k-1) = \sum_{k=a}^n f(k) - \sum_{k=a}^n (-1)^k f(k), \quad n \geq a+1 \quad (2.4)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} x^{2k} = \frac{1}{2} \frac{(1+x)^n - (1-x)^n}{x}, \quad n \geq 1 \quad (2.5)$$

2.1.1 Bifurcation Formulas

Bifurcation Formula

$$\sum_{k=a}^n f(k) = \sum_{k=\lceil \frac{a+1}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} f(2k) + \sum_{k=\lceil \frac{a+2}{2} \rceil}^{\lfloor \frac{n+1}{2} \rfloor} f(2k-1), \quad n \geq a+1 \quad (2.6)$$

Let $i = \sqrt{-1}$ in the following two equations.

$$\sum_{k=0}^n i^{k^2} f(k) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f(2k) + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} f(2k+1), \quad n \geq 1 \quad (2.7)$$

$$\sum_{k=0}^n (-1)^{\lfloor \frac{k}{2} \rfloor} i^{k^2} f(k) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k f(2k) + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k f(2k+1), \quad n \geq 1 \quad (2.8)$$

Generalized Bifurcation Formulas

$$\sum_{k=0}^{rn-1} f(k) = \sum_{j=0}^{r-1} \sum_{k=0}^{n-1} f(rk + j), \quad r \geq 1 \quad (2.9)$$

$$\sum_{k=a}^n f(k) = \sum_{j=0}^{r-1} \sum_{k=\lceil \frac{a+r-1-j}{r} \rceil}^{\lfloor \frac{n-j}{r} \rfloor} f(rk + j), \quad n - a + 1 \geq r \quad (2.10)$$

$$\sum_{k=a}^n f(k) = \sum_{k=\lceil \frac{a+2}{3} \rceil}^{\lfloor \frac{n}{3} \rfloor} f(3k) + \sum_{k=\lceil \frac{a+1}{3} \rceil}^{\lfloor \frac{n-1}{3} \rfloor} f(3k + 1) + \sum_{k=\lceil \frac{a}{3} \rceil}^{\lfloor \frac{n-2}{3} \rfloor} f(3k + 2), \quad n \geq a + 2 \quad (2.11)$$

$$\sum_{k=a}^n f(k) = \sum_{k=\lceil \frac{a+3}{4} \rceil}^{\lfloor \frac{n}{4} \rfloor} f(4k) + \sum_{k=\lceil \frac{a+2}{4} \rceil}^{\lfloor \frac{n-1}{4} \rfloor} f(4k + 1) + \sum_{k=\lceil \frac{a+1}{4} \rceil}^{\lfloor \frac{n-2}{4} \rfloor} f(4k + 2) + \sum_{k=\lceil \frac{a}{4} \rceil}^{\lfloor \frac{n-3}{4} \rfloor} f(4k + 3), \quad (2.12)$$

where $n \geq a + 3$.

Alternating Bifurcation Formula

$$\sum_{k=a}^n (-1)^k f(k) = \sum_{k=\lceil \frac{a+1}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} f(2k) - \sum_{k=\lceil \frac{a+2}{2} \rceil}^{\lfloor \frac{n+1}{2} \rfloor} f(2k - 1), \quad n \geq a + 1 \quad (2.13)$$

Generalized Alternating Bifurcation Formula

$$\sum_{k=a}^n (-1)^k f(k) = \sum_{j=0}^{r-1} \sum_{k=\lceil \frac{a+r-1-j}{r} \rceil}^{\lfloor \frac{n-j}{r} \rfloor} (-1)^{rk+j} f(rk + j), \quad n \geq a + r - 1 \quad (2.14)$$

$$\sum_{k=a}^n (-1)^k f(k) = \sum_{k=\lceil \frac{a+2}{3} \rceil}^{\lfloor \frac{n}{3} \rfloor} (-1)^k f(3k) - \sum_{k=\lceil \frac{a+1}{3} \rceil}^{\lfloor \frac{n-1}{3} \rfloor} (-1)^k f(3k + 1) + \sum_{k=\lceil \frac{a}{3} \rceil}^{\lfloor \frac{n-2}{3} \rfloor} (-1)^k f(3k + 2), \quad (2.15)$$

where $n \geq a + 2$.

$$\sum_{k=a}^n (-1)^k f(k) = \sum_{k=\lceil \frac{a+3}{4} \rceil}^{\lfloor \frac{n}{4} \rfloor} f(4k) - \sum_{k=\lceil \frac{a+2}{4} \rceil}^{\lfloor \frac{n-1}{4} \rfloor} f(4k+1) + \sum_{k=\lceil \frac{a+1}{4} \rceil}^{\lfloor \frac{n-2}{4} \rfloor} f(4k+2) - \sum_{k=\lceil \frac{a}{4} \rceil}^{\lfloor \frac{n-3}{4} \rfloor} f(4k+3), \quad (2.16)$$

where $n \geq a + 3$.

2.1.2 Basic Telescoping Identities

$$\sum_{k=1}^n (f(k) - f(k+r)) = \sum_{k=1}^r (f(k) - f(k+n)) \quad (2.17)$$

Let $\Delta_{k,r} f(k) = \frac{f(k+r) - f(k)}{r}$. Then,

$$\frac{1}{n} \sum_{k=1}^n \Delta_{k,r} f(k) = \frac{1}{r} \sum_{k=1}^r \Delta_{k,n} f(k) \quad (2.18)$$

2.1.3 Greatest Integer Function Identities

$$\sum_{k=1}^n f(k) = \frac{1}{2} \sum_{k=1}^{2n} f\left(\left[\frac{k+1}{2}\right]\right) \quad (2.19)$$

$$\sum_{k=1}^n f(k) = \frac{1}{2} \sum_{k=1}^{2n-1} f\left(\left[\frac{k}{2}\right] + 1\right) \quad (2.20)$$

$$\sum_{k=1}^n (-1)^{k-1} f(k) = \frac{1}{2} \sum_{k=1}^{2n-1} (-1)^{\lfloor \frac{k}{2} \rfloor} f\left(\left[\frac{k}{2}\right] + 1\right) \quad (2.21)$$

2.2 Expansions of $(1+i)^n$ and $(1-i)^n$

Remark 2.2 In Section 2.2, we let $i = \sqrt{-1}$.

Variation of Bifurcation Formula

$$\sum_{k=a}^n i^k f(k) = \sum_{k=\lceil \frac{a+1}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} (-1)^k f(2k) + i \sum_{k=\lceil \frac{a+2}{2} \rceil}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} f(2k-1) \quad (2.22)$$

Variation of Alternating Bifurcation Formula

$$\sum_{k=a}^n (-i)^k f(k) = \sum_{k=\lceil \frac{a+1}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} (-1)^k f(2k) - i \sum_{k=\lceil \frac{a+2}{2} \rceil}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} f(2k-1) \quad (2.23)$$

2.2.1 Expansion of $(1+i)^n$

$$(1+i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} - i \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n}{2k-1} \quad (2.24)$$

2.2.2 Expansions Involving $(1-i)^n$

$$(1-i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} + i \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n}{2k-1} \quad (2.25)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} = \frac{(1+i)^n + (1-i)^n}{2} \quad (2.26)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} = (\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right) \quad (2.27)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} = (-1)^n 2^{2n} \quad (2.28)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{2k} = (-1)^{\lfloor \frac{n}{2} \rfloor} (1 + (-1)^n) 2^{n-1} \quad (2.29)$$

$$\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \binom{n}{2k-1} = \frac{(1+i)^n - (1-i)^n}{2i} \quad (2.30)$$

$$\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \binom{n}{2k-1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{k-1} \binom{n}{2k+1} = (\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right) \quad (2.31)$$

$$\sum_{k=0}^{2n-1} (-1)^k \binom{4n}{2k+1} = 0 \quad (2.32)$$

$$\sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} = (-1)^{\lfloor \frac{n}{2} \rfloor} (1 - (-1)^n) 2^{n-1} \quad (2.33)$$

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{2k+1} = (-1)^{\lfloor \frac{n}{2} \rfloor} 2^n \quad (2.34)$$

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{2k} = (-1)^{\lfloor \frac{n+1}{2} \rfloor} 2^n \quad (2.35)$$

2.2.3 Expansions of $(\cos(\frac{\pi}{3}) \pm i \sin(\frac{\pi}{3}))^n$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} 3^k = 2^n \cos\left(\frac{n\pi}{3}\right) \quad (2.36)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} 3^k = \frac{2^n \sqrt{3}}{3} \sin\left(\frac{n\pi}{3}\right) \quad (2.37)$$

2.2.4 Expansions of $(\cos(\frac{\pi}{6}) \pm i \sin(\frac{\pi}{6}))^n$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \frac{1}{3^k} = \left(\frac{2\sqrt{3}}{3}\right)^n \cos\left(\frac{n\pi}{6}\right) \quad (2.38)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \frac{1}{3^k} = \sqrt{3} \left(\frac{2\sqrt{3}}{3}\right)^n \sin\left(\frac{n\pi}{6}\right) \quad (2.39)$$

2.3 Index Shift Formula with Applications

2.3.1 Index Shift Formula

Index Shift Formula: Version 1

$$\sum_{k=a}^n f(k) = \sum_{k=0}^{n-a} f(n-k), \quad a \geq 0 \quad (2.40)$$

Index Shift Formula: Version 2

$$\sum_{k=a}^n f(k) = \sum_{k=a}^n f(a+n-k), \quad a \geq 0 \quad (2.41)$$

2.3.2 Applications of Index Shift Formula

$$\sum_{k=0}^n \binom{2n}{k} = 2^{2n-1} + \frac{1}{2} \binom{2n}{n} \quad (2.42)$$

$$\sum_{k=0}^n \binom{2n}{k} = 2^{2n-1} + \binom{2n-1}{n}, \quad n \geq 1 \quad (2.43)$$

$$\sum_{k=1}^n \binom{2n}{n-k} = \sum_{k=1}^n \binom{2n}{n+k} = 2^{2n-1} - \frac{1}{2} \binom{2n}{n} \quad (2.44)$$

$$\sum_{k=1}^n \binom{2n}{n-k} = \sum_{k=1}^n \binom{2n}{n+k} = 2^{2n-1} - \binom{2n-1}{n}, \quad n \geq 1 \quad (2.45)$$

$$\sum_{k=0}^n \binom{2n-1}{k} = 2^{2n-2} + \binom{2n-1}{n}, \quad n \geq 1 \quad (2.46)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{2k} = 2^{2n-2} + \frac{1+(-1)^n}{2} \binom{2n-1}{n}, \quad (2.47)$$

otherwise, if $n = 0$, the sum equals 1.

$$\sum_{k=0}^n \binom{4n}{2k} = 2^{4n-2} + \binom{4n-1}{2n} \quad (2.48)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{2n}{2k+1} = 2^{2n-2} + \frac{1 - (-1)^n}{2} \binom{2n-1}{n}, \quad n \geq 1 \quad (2.49)$$

$$\sum_{k=0}^{n-1} \binom{4n}{2k+1} = 2^{4n-2} = 4^{2n-1}, \quad n \geq 1 \quad (2.50)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n+2k} = 2^{2n-2} + \frac{1}{2} \binom{2n}{n} \quad (2.51)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n+2k} = 2^{2n-2} + \binom{2n-1}{n} \quad (2.52)$$

$$\sum_{k=0}^n \binom{4n}{2n-2k} = \sum_{k=0}^n \binom{4n}{2n+2k} = 2^{4n-2} + \frac{1}{2} \binom{4n}{2n} \quad (2.53)$$

$$\sum_{k=0}^n \binom{4n}{2k} = \sum_{k=0}^n \binom{4n}{4n-2k} = 2^{4n-2} + \frac{1}{2} \binom{4n}{2n} \quad (2.54)$$

Variation of Index Shift Formula

$$\sum_{k=1}^n f(k) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (f(k) + f(n-k+1)) + \frac{1 - (-1)^n}{2} f\left(\left\lceil \frac{n+1}{2} \right\rceil\right), \quad n \geq 2 \quad (2.55)$$

$$\sum_{k=0}^n f(k) = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (f(k) + f(n-k)) + \frac{1 + (-1)^n}{2} f\left(\left\lfloor \frac{n}{2} \right\rfloor\right), \quad n \geq 1 \quad (2.56)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{k} = 2^{n-1} - \frac{1 + (-1)^n}{4} \binom{n}{\lfloor \frac{n}{2} \rfloor}, \quad n \geq 1 \quad (2.57)$$

$$\sum_{k=0}^n \binom{2n+1}{k} = 2^{2n}, \quad n \geq 0 \quad (2.58)$$

Application of Index Shift Formula with -1 Transformation

$$\sum_{k=n}^{r+n} \binom{n-1-k}{k} f(k) = (-1)^n \sum_{k=0}^r (-1)^k \binom{n+2k}{k} f(k+n), \quad r, n \geq 0 \quad (2.59)$$

2.3.3 Iterated Index Shift Formula

$$\sum_{k=0}^n \sum_{j=0}^k f(k, j) = \sum_{j=0}^n \sum_{k=0}^j f(n-k, n-j) \quad (2.60)$$

2.4 Series Properties of Periodic Functions

In the following identity, suppose $f(x) = f(\pi - x)$.

$$\sum_{k=0}^{n-1} f\left(\frac{2k+1}{4n}\pi\right) = \sum_{k=0}^{n-1} f\left(\frac{4k+1}{4n}\pi\right), \quad n \geq 1 \quad (2.61)$$

In the following identity, suppose $f(x) = -f(\pi - x)$.

$$\sum_{k=0}^{n-1} (-1)^k f\left(\frac{2k+1}{4n}\pi\right) = \sum_{k=0}^{n-1} f\left(\frac{4k+1}{4n}\pi\right), \quad n \geq 1 \quad (2.62)$$

In the following identity suppose $f(x) = -f(2\pi - x)$

$$\sum_{k=0}^{2n-1} (-1)^k f\left(\frac{2k+1}{4n}\pi\right) = \sum_{k=0}^{2n-1} f\left(\frac{4k+1}{4n}\pi\right), \quad n \geq 1 \quad (2.63)$$

3 Calculus Operations on Series

3.1 Four Basic Integral Formulas

Remark 3.1 In Section 3.1, we assume a and b are nonnegative integers. We assume p is a positive integer. Furthermore, we assume that $a_b \leq a_{b+1} \leq \dots \leq a_n \leq a_{n+1} \leq a_{n+2} \leq a_{n+3}$. Lastly, recall the $[x]$ is the greatest integer in x .

3.1.1 First Integral Formula

$$\sum_{r=b}^n \varphi(r) \int_{a_r}^{a_{r+1}} f(x) dx = \varphi(b) \int_{a_b}^{a_{b+1}} f(x) dx + \sum_{r=b+1}^n (\varphi(r) - \varphi(r-1)) \int_{a_r}^{a_{r+1}} f(x) dx \quad (3.1)$$

Applications of First Integral Formula

$$\sum_{r=a}^n (\varphi(r) - \varphi(r-1))(n+1-r) = \sum_{r=a}^n \varphi(r) - (n-a+1)\varphi(a-1) \quad (3.2)$$

$$\sum_{r=a}^n ((-1)^r - (-1)^{r-1})(n-r+1) = \sum_{r=a}^n (-1)^r - (-1)^{a-1}(n-a+1) \quad (3.3)$$

$$\sum_{r=0}^n (-1)^r (n_r + 1) = \frac{(-1)^n + 2n + 3}{4} \quad (3.4)$$

$$\sum_{r=0}^n \left(\binom{n}{r} - \binom{n}{r-1} \right) (n-r+1) = \sum_{r=0}^n \binom{n}{r} \quad (3.5)$$

$$\sum_{r=0}^n \binom{n}{r} r = n2^{n-1} \quad (3.6)$$

$$\sum_{r=1}^n f(r) = nf(n) - \sum_{r=1}^{n-1} r(f(r+1) - f(r)) \quad (3.7)$$

$$\sum_{k=1}^n \frac{f(k)}{k} = \frac{nf(n+1)}{n+1} - \sum_{k=1}^n k \left(\frac{f(k+1)}{k+1} - \frac{f(k)}{k} \right) \quad (3.8)$$

$$\sum_{r=1}^n r! = nn! - \sum_{r=1}^{n-1} r^2 r! \quad (3.9)$$

$$\prod_{k=1}^n \left(\frac{f(k+1)}{f(k)} \right)^k = \frac{(f(n+1))^n}{\prod_{k=1}^n f(k)} \quad (3.10)$$

$$\sum_{k=1}^n (k^p - (k-1)^p) f(k) = n^p f(n+1) - \sum_{k=1}^n k^p (f(k+1) - f(k)) \quad (3.11)$$

$$\sum_{k=1}^n (k^p - (k-1)^p) f(k) = \sum_{j=1}^p (-1)^j \binom{p}{j} \sum_{k=1}^n k^{p-j} f(k) \quad (3.12)$$

$$\prod_{k=1}^n \left(\frac{f(k+1)}{f(k)} \right)^{k^p} = (f(n+1))^{n^p} \prod_{k=1}^n (f(k))^{(k-1)^p - k^p} \quad (3.13)$$

3.1.2 Second Integral Formula

$$\sum_{r=b}^n \int_{a_r}^{a_{r+1}} f(x) dx = \int_{a_b}^{a_{n+1}} f(x) dx \quad (3.14)$$

Applications of Second Integral Formula

$$\sum_{r=0}^n \frac{2r+3}{(r+1)^2(r+2)^2} = 1 - \frac{1}{(n+2)^2} \quad (3.15)$$

$$\sum_{r=0}^n \frac{2r+5}{(r+2)^2(r+3)^2} = \frac{1}{4} - \frac{1}{(n+3)^2} \quad (3.16)$$

$$\sum_{r=0}^n \frac{2r+2a+1}{(r+a)^2(r+a+1)^2} = \frac{1}{a^2} - \frac{1}{(n+a+1)^2}, \quad a \geq 1 \quad (3.17)$$

$$\sum_{r=0}^n \frac{3r^2+9r+7}{(r+1)^3(r+2)^3} = 1 - \frac{1}{(n+1)^3} \quad (3.18)$$

$$\sum_{r=1}^{\infty} \frac{3r^2 + 3r + 1}{r^3(r+1)^3} = 1 \quad (3.19)$$

$$\sum_{r=0}^n (r^2 + 2r)(r!)^2 = (n+1)! - 1 \quad (3.20)$$

$$\sum_{r=0}^n ((r+1)^p - 1)(r!)^p = ((n+1)!)^p - 1 \quad (3.21)$$

$$\sum_{r=0}^n rr! = (n+1)! - 1 \quad (3.22)$$

$$\sum_{r=0}^n \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!} \quad (3.23)$$

$$\sum_{r=0}^n \frac{r^2 + 2r}{(r+1)^2(r!)^2} = 1 - \frac{1}{((n+1)!)^2} \quad (3.24)$$

$$\sum_{r=0}^n \left(1 - \frac{1}{(r+1)^p}\right) \frac{1}{(r!)^p} = 1 - \frac{1}{((n+1)!)^p} \quad (3.25)$$

$$\sum_{r=0}^n \left(1 - \frac{1}{\sqrt{r+1}}\right) \frac{1}{\sqrt{r!}} = 1 - \frac{1}{(\sqrt{(n+1)!})} \quad (3.26)$$

$$\sum_{r=1}^{\infty} \frac{r - \sqrt{r}}{r\sqrt{(r-1)!}} = 1 \quad (3.27)$$

3.1.3 Third Integral Formula

$$\sum_{r=b}^n (-1)^r \int_{a_r}^{a_{r+2}} f(x) dx = (-1)^b \int_{a_b}^{a_{b+1}} f(x) dx + (-1)^n \int_{a_{n+1}}^{a_{n+2}} f(x) dx \quad (3.28)$$

Applications of Third Integral Formula

$$\sum_{r=b}^n (-1)^r (F(r+2) - F(r)) = (-1)^b (F(b+1) - F(b)) + (-1)^n (F(n+2) - F(n+1)) \quad (3.29)$$

$$\sum_{k=0}^n (-1)^k (k^2 + 3k + 1)k! = (-1)^n (n+1)(n+1)! \quad (3.30)$$

$$\sum_{k=0}^n (-1)^k \frac{1}{(k+1)(k+3)} = \frac{1}{4} + \frac{(-1)^n}{2(n+2)(n+3)} \quad (3.31)$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)(k+3)} = \frac{1}{4} \quad (3.32)$$

$$\sum_{k=0}^n (-1)^k \frac{k^2 + 3k + 1}{(k+2)!} = (-1)^n \frac{n+1}{(n+2)!} \quad (3.33)$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{k^2 + 3k + 1}{(k+2)!} = 0 \quad (3.34)$$

3.1.4 Fourth Integral Formula

$$\sum_{r=0}^n (-1)^r \int_{a_r}^{a_{r+3}} f(x) dx = \sum_{r=2}^n (-1)^r \int_{a_r}^{a_{r+1}} f(x) dx + \int_{a_0}^{a_1} f(x) dx + (-1)^n \int_{a_{n+2}}^{a_{n+3}} f(x) dx \quad (3.35)$$

Applications of Fourth Integral Formula

$$\begin{aligned} \sum_{r=0}^n (-1)^r (F(r+3) - F(r)) &= \sum_{r=2}^n (-1)^r (F(r+1) - F(r)) + F(1) - F(0) \\ &\quad + (-1)^n (F(n+3) - F(n+2)) \end{aligned} \quad (3.36)$$

$$\sum_{r=0}^n (-1)^r = \frac{1 + (-1)^n}{2} \quad (3.37)$$

$$\sum_{r=0}^n (-1)^r r = \frac{(2n+1)(-1)^n - 1}{4} = (-1)^n \left[\frac{n+1}{2} \right] \quad (3.38)$$

$$\sum_{r=0}^n (-1)^r r^2 = (-1)^n \frac{n^2 + n}{2} \quad (3.39)$$

3.2 Three Integration by Parts Formulas

3.2.1 First Integration by Parts Formula

$$\sum_{k=1}^n f(k)(\varphi(k) - \varphi(k-1)) = f(n)\varphi(n) - f(1)\varphi(0) - \sum_{k=1}^{n-1} \varphi(k)(f(k+1) - f(k)) \quad (3.40)$$

Applications of First Integration by Parts Formula

$$\sum_{k=1}^n \frac{f(k)}{k(k+1)} = f(1) - \frac{f(n)}{n+1} - \sum_{k=1}^{n-1} \frac{f(k) - f(k+1)}{k+1} \quad (3.41)$$

$$\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3(k+1)} \quad (3.42)$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} = 1 - \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} = 2 - \frac{\pi^2}{6} \quad (3.43)$$

3.2.2 Second Integration by Parts Formula

Assume a is a nonnegative integer. Let $S_n = \sum_{i=0}^n a_i$.

$$\sum_{k=a+1}^n a_k b_k = \sum_{k=a+1}^n S_k (b_k - b_{k+1}) + S_n b_{n+1} - S_a b_{a+1} \quad (3.44)$$

3.2.3 Third Integration by Parts Formula

$$\int \prod_{i=0}^{n-1} u_i du_n = \prod_{i=1}^n u_i - \sum_{k=1}^{n-1} \int \left(\prod_{i=1}^n u_i \right) \frac{1}{u_k} du_k \quad (3.45)$$

3.3 Taylor's Theorem

Remark 3.2 Let $f(x) = \sum_{i=0}^n a_i x^i$, where the a_i are independent of x . Let $f^{(k)}(x)$ denote the k^{th} derivative of $f(x)$. Let $f^{(k)}(y)$ be the k^{th} derivative with respect to x evaluated at y .

Taylor's Theorem

$$f(x) = \sum_{k=0}^n \frac{(x-y)^k}{k!} f^{(k)}(y) \quad (3.46)$$

Two Variations of Taylor's Theorem

$$f(x+y) = \sum_{k=0}^n \frac{x^k}{k!} f^{(k)}(y) \quad (3.47)$$

$$f(x) = \sum_{k=0}^n \frac{x^k}{k!} f^{(k)}(0) \quad (3.48)$$

3.4 Taylor's Theorem for Real Valued Functions of Several Variables

Let φ be a real valued function of n variables, say (x_1, x_2, \dots, x_n) . Let $\frac{\partial^{j_i}}{\partial x_i^{j_i}}$ be the partial derivative (with respect to the variable x_i) of $\varphi(x_1, \dots, x_n)$ taken j_i times. Then,

$$\begin{aligned} &\varphi(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n) = \\ &\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\forall j, 0 \leq j_i \leq k, \\ \sum_{i=1}^n j_i = k}} \frac{k!}{j_1! j_2! \dots j_n!} \left(h_1^{j_1} \frac{\partial^{j_1}}{\partial x_1^{j_1}} h_2^{j_2} \frac{\partial^{j_2}}{\partial x_2^{j_2}} \dots h_n^{j_n} \frac{\partial^{j_n}}{\partial x_n^{j_n}} \right) \varphi(x_1, \dots, x_n) \end{aligned} \quad (3.49)$$

3.5 Leibnitz Formula: Generalized Product Rule for Differentiation

Remark 3.3 In Section 3.5, we assume u and v be r -times differentiable functions of x , where $r \geq 0$ and integral.

Leibnitz Formula

$$\frac{d^r}{dx^r}(uv) = \sum_{k=0}^r \binom{r}{k} \frac{d^{r-k} u}{dx^{r-k}} \frac{d^k v}{dx^k} \quad (3.50)$$

3.5.1 Applications of Leibnitz Formula

$$\sum_{k=0}^n \binom{k}{r} x^k = \frac{x^r}{(1-x)^{r+1}} - x^{n+1} \sum_{k=0}^{\infty} \binom{n+1+k}{r} x^k, \quad |x| < 1 \quad (3.51)$$

$$\sum_{k=1}^{\infty} \binom{2n+k}{n} \frac{1}{2^k} = 2^{2n} \quad (3.52)$$

$$\sum_{k=0}^n \binom{k}{r} x^k = \frac{x^r}{(1-x)^{r+1}} + \binom{n}{r} x^n - \frac{n!}{r!} x^n \sum_{k=0}^r \binom{r}{k} \frac{k!}{(k-r+n)!} \frac{x^k}{(1-x)^{k+1}} \quad (3.53)$$

$$\sum_{k=0}^{2n} \binom{k}{n} x^k = \frac{x^n}{(1-x)^{n+1}} + \binom{2n}{n} x^{2n} - x^{2n} \sum_{k=0}^n \binom{2n}{k+n} \frac{x^k}{(1-x)^{k+1}} \quad (3.54)$$

$$\sum_{k=0}^n \binom{k+n}{n} x^k = \frac{1}{(1-x)^{n+1}} + \binom{2n}{n} x^n - \sum_{k=0}^n \binom{2n}{k+n} \frac{x^{k+n}}{(1-x)^{k+1}} \quad (3.55)$$

3.6 Three Versions of the Generalized Chain Rule

Remark 3.4 In Section 3.6, we will let D_z represent differentiation with respect to z . Hence, $D_z^n f(x)$ is the n^{th} derivative of $f(x)$ with respect to z , i.e. $D_z^n f(x) = D_z D_z \dots D_z f(x)$, where the product contains n factors. We will let D_x represent differentiation with respect to x . We also assume that x is a function of z , i.e. $x = x(z)$. Finally, we let, α , unless otherwise specified, denote a nonnegative integer.

3.6.1 Version 1: Hoppe Form of Generalized Chain Rule

$$D_z^n f(x) = \sum_{\alpha=0}^n D_x^\alpha f(x) \frac{(-1)^\alpha}{\alpha!} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} x^{\alpha-j} D_z^n x^j \quad (3.56)$$

Applications of Version 1

$$D_z^n f(x)|_{z=a+bx} = \frac{1}{b^n} D_x^n f(x), \quad b \neq 0 \quad (3.57)$$

Remark 3.5 *In the following identity, α is any real number. Also, we assume $u = u(x)$.*

$$D_x^n u^\alpha = \sum_{k=0}^n (-1)^k \binom{\alpha}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} u^{\alpha-j} D_x^n u^j \quad (3.58)$$

Remark 3.6 *In the following identity, let $x = e^z$. Then, $D_z^n = (xD_x)^n$.*

Gunnert's Formula

$$D_z^n f(x) = (xD_x)^n f(x) = \sum_{\alpha=0}^n D_x^\alpha f(x) e^{\alpha z} \frac{(-1)^\alpha}{\alpha!} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} j^n \quad (3.59)$$

Derivatives of Reciprocal Functions

$$D_z^n \left(\frac{1}{x} \right) = \sum_{\alpha=0}^n \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{1}{x^{j+1}} D_z^n x^j \quad (3.60)$$

$$D_z^n \left(\frac{1}{x} \right) = \sum_{j=0}^n (-1)^j \binom{n+1}{j+1} \frac{1}{x^{j+1}} D_z^n x^j \quad (3.61)$$

$$D_x^n \left(\frac{1}{f(x)} \right) = \sum_{\alpha=0}^n \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{1}{(f(x))^{j+1}} D_x^n (f(x))^j \quad (3.62)$$

$$D_x^n \left(\frac{1}{f(x)} \right) = \sum_{j=0}^{\alpha} (-1)^j \binom{n+1}{j+1} \frac{1}{(f(x))^{j+1}} D_x^n (f(x))^j \quad (3.63)$$

Remark 3.7 *In the following identity, assume $f(x) = \sum_{i=0}^{\infty} a_i x^i$, with $f(0) = a_0 \neq 0$. Note that all a_i are independent of x . Furthermore, assume for a nonnegative integer β , $(f(x))^\beta = \sum_{j=0}^{\infty} b_j^{(\beta)} x^j$. Once again, all $b_j^{(\beta)}$ are independent of x .*

$$\frac{1}{f(x)} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=0}^j (-1)^k \binom{j+1}{k+1} \frac{b_j^{(k)}}{a_0^{k+1}} \quad (3.64)$$

Remark 3.8 In the following two identities, both u and x are functions of z .

$$D_z^n \left(\frac{u}{x} \right) = \sum_{k=0}^n \binom{n}{k} D_z^{n-k} u \sum_{\alpha=0}^k \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{1}{x^{j+1}} D_z^k x^j \quad (3.65)$$

$$D_z^n \left(\frac{u}{x} \right) = \sum_{k=0}^n \binom{n}{k} D_z^{n-k} u \sum_{j=0}^k \binom{k+1}{j+1} \frac{(-1)^j}{x^{j+1}} D_z^k x^j \quad (3.66)$$

Remark 3.9 In the following two identities, assume f is any n -times differentiable function. Also assume a is independent of x

$$\frac{(a-x)^{n+1}}{n!} D_x^n \left(\frac{f(x)}{a-x} \right) = \sum_{k=0}^n \frac{(a-x)^k}{k!} D_x^k f(x) \quad (3.67)$$

$$\frac{(a-x)^{n+1}}{n!} D_x^n \left(\frac{f(a) - f(x)}{a-x} \right) = f(a) - \sum_{k=0}^n \frac{(a-x)^k}{k!} D_x^k f(x) \quad (3.68)$$

Remark 3.10 In the following identity, due to G. H. Halphen, we assume f and ϕ are n -times differentiable functions. We also let $\phi^{(k)} \left(\frac{1}{x} \right)$ denote the k^{th} derivative of $\phi \left(\frac{1}{x} \right)$ with respect to $\frac{1}{x}$.

$$D_x^n \left(f(x) \phi \left(\frac{1}{x} \right) \right) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{x^k} \phi^{(k)} \left(\frac{1}{x} \right) D_x^{n-k} \left(\frac{f(x)}{x^k} \right) \quad (3.69)$$

Remark 3.11 *The following identity is a generalization of the Version 1 due to R. Most. He assumes that both f and ϕ are $(n + m)$ -times differentiable functions, where m is an arbitrary nonnegative integer.*

$$D_z^n f(x) = \sum_{k=0}^{n+m} \frac{(-1)^k}{k!} D_x^k (f(x)\phi(x)) \sum_{j=0}^k (-1)^j \binom{k}{j} x^{k-j} D_z^n \left(\frac{x^j}{\phi(x)} \right) \quad (3.70)$$

Remark 3.12 *In the following identity, due to R. Most, α and β are arbitrary real numbers.*

$$D_z^n x^\alpha = (\alpha + \beta) \binom{n + m - \alpha - \beta}{n + m} \sum_{j=0}^{n+m} (-1)^j \binom{n + m}{j} \frac{x^{\alpha+\beta-j}}{\alpha + \beta - j} D_z^n x^{j-\beta} \quad (3.71)$$

3.6.2 Version 2: Operator Form of Generalized Chain Rule

$$D_z^n f(x) = \sum_{j=0}^n A_j^n(z) D_x^j f(x), \quad (3.72)$$

where $A_j^n(z)$ are independent of f and calculated by

$$A_j^n(z) = \frac{1}{j!} D_t^j (e^{-tx} D_z^n e^{tx}) |_{t=0} \quad (3.73)$$

3.6.3 Version 3: Faa di Bruno's Formula for the Generalized Chain Rule

$$D_z^n f(x) = \sum_{k=1}^n D_x^k f(x) \frac{1}{k!} \sum \frac{n!}{j_1! j_2! \dots j_\alpha!} \left(\frac{1}{k_1!} D_z^{k_1} x \right)^{j_1} \dots \left(\frac{1}{k_\alpha!} D_z^{k_\alpha} x \right)^{j_\alpha}, \quad (3.74)$$

where the inner sum is extended over all partitions such that $\sum_i^\alpha j_i = k$ and $\sum_{i=1}^\alpha j_i k_i = n$.

4 Iterative Series

Remark 4.1 *In this chapter, recall that $[x]$ is the greatest integer in x .*

4.1 First Example of an Iterative Series

$$\sum_{r=0}^n \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} A_{r,k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{2k}^n A_{r,k} \quad (4.1)$$

4.1.1 Applications of the First Iterative Series

$$\sum_{r=0}^n \lfloor \frac{r}{2} \rfloor = n \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor^2 \quad (4.2)$$

$$\sum_{r=0}^n \lfloor \frac{r}{2} \rfloor f(r) = \lfloor \frac{n}{2} \rfloor \sum_{r=0}^n f(r) - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=0}^{2k-1} f(r) \quad (4.3)$$

$$\sum_{r=0}^n (-1)^r \lfloor \frac{n}{2} \rfloor = \frac{1 + (-1)^n}{2} \lfloor \frac{n}{2} \rfloor \quad (4.4)$$

$$\sum_{r=0}^n \binom{n}{r} \lfloor \frac{r}{2} \rfloor = \lfloor \frac{n}{2} \rfloor 2^n - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=0}^{2k-1} \binom{n}{r} \quad (4.5)$$

$$\sum_{r=0}^n \lfloor \frac{r}{2} \rfloor^2 = \lfloor \frac{n}{2} \rfloor \frac{1 + 3n \lfloor \frac{n}{2} \rfloor - 4 \lfloor \frac{n}{2} \rfloor^2}{3} \quad (4.6)$$

Remark 4.2 *In the following identity, assume $\{a_n\}_{n=0}^{\infty}$ is a sequence which obeys the Fibonacci recurrence, i.e. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$*

$$\sum_{r=0}^n \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r-k}{k} = \sum_{r=0}^n a_r = a_{n+2} - 1 \quad (4.7)$$

$$\sum_{r=0}^n 2^r \cos\left(\frac{r\pi}{3}\right) = \frac{2^{n+1}\sqrt{3}}{3} \sin\left(\frac{(n+1)\pi}{3}\right) \quad (4.8)$$

Generalization of First Iterative Series

$$\sum_{k=a}^n \sum_{i=a}^{\lfloor \frac{k}{r} \rfloor} A_{i,k} = \sum_{i=a}^{\lfloor \frac{n}{r} \rfloor} \sum_{ri}^n A_{i,k}, \quad (4.9)$$

where r and a are integers such that $r \geq 1$ and $a \geq 0$.

4.2 Second Example of an Iterative Series

$$\sum_{k=1}^n \sum_{i=1}^{\lfloor \sqrt{k} \rfloor} A_{i,k} = \sum_{i=1}^{\lfloor \sqrt{n} \rfloor} \sum_{k=i^2}^n A_{i,k} \quad (4.10)$$

4.2.1 Applications of the Second Iterative Series

$$\sum_{k=1}^n \lfloor \sqrt{k} \rfloor f(k) = \sum_{i=1}^{\lfloor \sqrt{n} \rfloor} \sum_{k=i^2}^n f(k) \quad (4.11)$$

$$\sum_{k=1}^n \lfloor \sqrt{k} \rfloor = \lfloor \sqrt{n} \rfloor \left(n + 1 - \frac{2\lfloor \sqrt{n} \rfloor^2 + 3\lfloor \sqrt{n} \rfloor + 1}{6} \right) \quad (4.12)$$

$$\sum_{k=1}^n \frac{\lfloor \sqrt{k} \rfloor}{2^k} = 2 \sum_{i=1}^{\lfloor \sqrt{n} \rfloor} \frac{1}{2^{i^2}} - \frac{\lfloor \sqrt{n} \rfloor}{2^n} \quad (4.13)$$

$$\sum_{k=1}^{\infty} \frac{\lfloor \sqrt{k} \rfloor}{2^k} = \sum_{i=1}^{\infty} \frac{1}{2^{i^2-1}} \quad (4.14)$$

4.3 Third Example of an Iterative Series

$$\sum_{i=1}^n \sum_{k=1}^{2^i-1} A_{i,k} = \sum_{k=1}^{2^n-1} \sum_{i=1+\lceil \log_2 k \rceil}^n A_{i,k} \quad (4.15)$$

4.3.1 Applications of the Third Iterative Series

$$\sum_{k=1}^{2^n-1} \lceil \log_2 k \rceil f(k) = n \sum_{k=1}^{2^n-1} f(k) - \sum_{i=1}^n \sum_{k=1}^{2^i-1} f(k) \quad (4.16)$$

$$\sum_{k=1}^{2^n-1} \lceil \log_2 k \rceil = (n-2)2^n + 2 \quad (4.17)$$

$$\sum_{k=1}^{2^n-1} \lceil \log_2(2k) \rceil = (n-1)2^n + 1 \quad (4.18)$$

$$\sum_{k=1}^{2^n-1} \lceil \log_2(2k-1) \rceil = (n-2)2^{n-1} + 1 \quad (4.19)$$

$$\sum_{k=1}^{2^n-1} (-1)^k \lceil \log_2 k \rceil = 0 \quad (4.20)$$

$$\sum_{k=1}^{2^n} (-1)^k \lceil \log_2 k \rceil = n \quad (4.21)$$

$$\sum_{i=1}^n \sum_{k=1}^{2^i-1} f(i) = \sum_{k=1}^{2^n-1} \sum_{i=1+\lceil \log_2 k \rceil}^n f(i) \quad (4.22)$$

4.4 Standard Interchange Formula for Iterative Series

Remark 4.3 In Section 4.4, we assume a is a nonnegative integer with $a \leq n$.

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{i=a}^n \sum_{k=i}^n A_{i,k} \quad (4.23)$$

4.4.1 Variations of Standard Interchange Formula

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{k=0}^{n-a} \sum_{i=a}^{n-k} A_{i,n-k} \quad (4.24)$$

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{i=a}^n \sum_{k=0}^{n-i} A_{i,k+i} \quad (4.25)$$

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{i=0}^{n-a} \sum_{k=n-i}^n A_{n-i,k} \quad (4.26)$$

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{i=0}^{n-a} \sum_{k=0}^i A_{n-i,k+n-i} \quad (4.27)$$

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{k=0}^{n-a} \sum_{i=k}^{n-a} A_{n-i,k+n-i} \quad (4.28)$$

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{k=a}^n \sum_{i=k-a}^{n-a} A_{n-i,k-a+n-i} \quad (4.29)$$

$$\sum_{k=a}^n \sum_{i=a}^k A_{i,k} = \sum_{k=a}^n \sum_{i=k}^n A_{n-i+a,k+n-i} \quad (4.30)$$

4.4.2 Applications of Standard Interchange Formula

$$\sum_{k=0}^n \sum_{i=0}^{2n-k} A_{i,k} = \sum_{i=0}^{2n} \sum_{k=0}^i A_{i-k,k} - \sum_{i=n+1}^{2n} \sum_{k=n+1}^i A_{i-k,k} \quad (4.31)$$

$$\sum_{k=a}^n \sum_{j=a}^k f(j) = \sum_{j=a}^n \sum_{k=j}^n f(j) \quad (4.32)$$

$$\sum_{j=a}^n jf(j) = (n+1) \sum_{j=a}^n f(j) - \sum_{k=a}^n \sum_{j=a}^k f(j) \quad (4.33)$$

$$\sum_{k=1}^n k \left[\frac{x + 2^{k-1}}{2^k} \right] = \sum_{k=1}^n \left[\frac{x}{2^k} \right] - (n+1) \left[\frac{x}{2^n} \right] + [x] \quad (4.34)$$

$$\sum_{k=0}^n k \binom{k}{m} = \frac{mn + n + m}{m+2} \binom{n+1}{m+1} \quad (4.35)$$

where m is a nonnegative integer

4.5 Fourth Example of an Iterative Series

Remark 4.4 In Section 4.5, we assume a and r are integers such that $0 \geq a \geq n$ and $r \geq 1$.

$$\sum_{k=a}^n \sum_{i=a}^{rk} A_{i,k} = \sum_{i=a}^{rn} \sum_{k=\lceil \frac{i+r-1}{r} \rceil}^n A_{i,k} \quad (4.36)$$

4.5.1 Applications of the Fourth Iterative Series

$$\sum_{k=1}^n \sum_{i=1}^{2k} A_{i,k} = \sum_{i=1}^{2n} \sum_{k=\lceil \frac{i+1}{2} \rceil}^n A_{i,k} \quad (4.37)$$

$$\sum_{k=1}^n \sum_{i=1}^{3k} A_{i,k} = \sum_{i=1}^{3n} \sum_{k=\lceil \frac{i+2}{3} \rceil}^n A_{i,k} \quad (4.38)$$

$$\sum_{i=1}^{rn} \left[\frac{i-1}{r} \right] = \frac{rn(n-1)}{2} \quad (4.39)$$

$$\sum_{k=0}^n \sum_{i=k}^{2k} A_{i,k} = \sum_{i=0}^{2n} \sum_{k=\lceil \frac{i+1}{2} \rceil}^{i-\lceil \frac{i}{n+1} \rceil(i-n)} A_{i,k} \quad (4.40)$$

$$\sum_{k=0}^{\infty} \sum_{i=k}^{2k} A_{i,k} = \sum_{i=0}^{\infty} \sum_{k=\lceil \frac{i+1}{2} \rceil}^i A_{i,k} \quad (4.41)$$

$$\sum_{k=0}^{\infty} \sum_{i=k}^{3k} A_{i,k} = \sum_{i=0}^{\infty} \sum_{k=\lceil \frac{i+2}{3} \rceil}^i A_{i,k} \quad (4.42)$$

$$\sum_{k=1}^{\infty} \sum_{i=k}^{2k} A_{i,k} = \sum_{i=1}^{\infty} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} A_{i,i-k} \quad (4.43)$$

4.6 Fifth Example of an Iterative Series

Remark 4.5 In Section 4.6, we assume p is a positive integer.

$$\sum_{k=1}^n \sum_{i=1}^{k^p} A_{i,k} = \sum_{i=1}^{n^p} \sum_{k=1+\lceil \sqrt[p]{i-1} \rceil}^n A_{i,k} \quad (4.44)$$

4.6.1 Applications of the Fifth Iterative Series

$$\sum_{k=1}^n \sum_{i=1}^{k^2} A_{i,k} = \sum_{i=1}^{n^2} \sum_{k=1+\lceil \sqrt{i-1} \rceil}^n A_{i,k} \quad (4.45)$$

$$\sum_{k=1}^n \sum_{i=1}^{k^3} A_{i,k} = \sum_{i=1}^{n^3} \sum_{k=1+\lceil \sqrt[3]{i-1} \rceil}^n A_{i,k} \quad (4.46)$$

$$\sum_{k=2}^{\infty} \sum_{i=2k}^{k^2} A_{i,k} = \sum_{i=2}^{\infty} \sum_{k=1+\lceil \sqrt{i-1} \rceil}^{\lfloor \frac{i}{2} \rfloor} A_{i,k} \quad (4.47)$$

4.7 Two Special Iterative Series

Remark 4.6 In the following identity, we define $!^{-1}$ as the inverse function of $!$. That is, $x! = n$ if and only if $!^{-1}n = x$. Furthermore, we assume $!^{-1}1 = 1$.

$$\sum_{k=2}^n \sum_{i=2}^{k!} A_{i,k} = \sum_{i=2}^{n!} \sum_{k=1+!^{-1}(i-1)}^n A_{i,k} \quad (4.48)$$

Remark 4.7 In the following identity, we assume $g(x)$ is a function such that $x^x = z$ if and only if $x = g(z)$.

$$\sum_{k=2}^n \sum_{i=2}^{k^2} A_{i,k} = \sum_{i=2}^{n^n} \sum_{k=1+[g(i-1)]}^n A_{i,k} \quad (4.49)$$

4.8 Iterations of the Hockey Stick Identity

Let

$$\sum_{(r)k_1} f(k_1) \equiv \sum_{k_r=0}^n \sum_{k_{r-1}=0}^{k_r} \sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_1=0}^{k_2} f(k_1)$$

be the r -fold iterated sum of $f(k_1)$.

Iterated Hockey Stick Identity

$$\sum_{(r)j} \binom{j}{k} = \binom{n+r}{k+r}, \quad r \geq 1 \quad n, k \geq 0 \quad (4.50)$$

4.9 Iterated Sums with Deleted Terms

$$\sum_{j=0}^n \sum_{i=0}^n A_{i,j} - \sum_{i=0}^n A_{i,i} = \sum_{j=0}^n \sum_{\substack{i=0 \\ i \neq j}}^n A_{i,j} = \sum_{i=0}^n \sum_{\substack{j=0 \\ j \neq i}}^n A_{i,j} \quad (4.51)$$

4.9.1 Applications of Deleted Terms Identity

Remark 4.8 In the following identities, let $A_{i,j} = u_i v_j$.

$$\sum_{j=0}^n v_j \sum_{\substack{i=0 \\ i \neq j}}^n u^i = \sum_{j=0}^n \sum_{i=0}^n u_i v_j - \sum_{i=0}^n u_i v_j \quad (4.52)$$

$$\sum_{j=0}^n \sum_{\substack{i=0 \\ i \neq j}}^n u^i = n \sum_{i=0}^n u_i \quad (4.53)$$

$$\sum_{k_r=0}^n \sum_{\substack{k_{r-1}=0 \\ k_r \neq k_{r-1}}}^n \dots \sum_{\substack{k_1=0 \\ k_1 \neq k_2}}^n u_{k_1} = n^{r-1} \sum_{j=0}^n u_j, \quad (4.54)$$

where the left hand side is an r -fold iterated sum for fixed positive integers r and n .

$$\prod_{j=0}^n \prod_{\substack{i=0 \\ i \neq j}}^n u_i = \left(\prod_{i=0}^n u_i \right)^n \quad (4.55)$$

$$\prod_{k_r=0}^n \prod_{\substack{k_{r-1}=0 \\ k_r \neq k_{r-1}}}^n \dots \prod_{\substack{k_1=0 \\ k_1 \neq k_2}}^n u_{k_1} = \left(\prod_{j=0}^n u_j \right)^{n^{r-1}} \quad (4.56)$$

where the left hand side is an r -iterated product, for r and n fixed positive integers.

$$\sum_{j=0}^n \sum_{\substack{i=0 \\ i \neq j}}^n (j+1)u_i = \frac{n^2 + 3n}{2} \sum_{i=0}^n u_i - \sum_{i=0}^n i u_i \quad (4.57)$$

$$\sum_{j=0}^n \sum_{\substack{i=0 \\ i \neq j, n-j}}^n u_i = (n-1) \sum_{i=0}^n u_i + \frac{(-1)^n + 1}{2} u_{\lfloor \frac{n}{2} \rfloor} \quad (4.58)$$

$$\sum_{j=0}^n \sum_{i=0}^n A_{i,j} - \sum_{i=0}^n A_{i,n-i} = \sum_{j=0}^n \sum_{\substack{i=0 \\ i \neq n-j}}^n A_{i,j} \quad (4.59)$$

5 Three Convolution Formulas for Finite Series

5.1 First Convolution Formula

$$\begin{aligned} \sum_{j=1}^n \sum_{i=1}^n f(i)\varphi(j) &= \left(\sum_{i=1}^n f(i) \right) \left(\sum_{j=1}^n \varphi(j) \right) \\ &= \sum_{k=1}^n f(k)\varphi(k) + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} (f(i)\varphi(i+j) + f(i+j)\varphi(i)), \quad n \geq 2 \end{aligned} \quad (5.1)$$

5.1.1 Applications of First Convolution Formula

$$\left(\sum_{i=0}^n A_i \right) \left(\sum_{j=0}^n B_j \right) = \sum_{k=0}^n A_k B_k + \sum_{r=1}^n \sum_{k=0}^{n-r} (A_k B_{k+r} + A_{k+r} B_k), \quad n \geq 1 \quad (5.2)$$

$$\left(\sum_{i=1}^n f(i) \right)^2 = \sum_{k=1}^n (f(k))^2 + 2 \sum_{r=1}^{n-1} \sum_{k=1}^{n-r} f(k)f(k+r), \quad n \geq 2 \quad (5.3)$$

$$\left(\sum_{k=1}^n \frac{1}{k!} \right)^2 = \sum_{k=1}^n \frac{1}{(k!)^2} + 2 \sum_{r=1}^{n-1} \sum_{k=1}^{n-r} \frac{1}{k!(k+r)!}, \quad n \geq 2 \quad (5.4)$$

$$\sum_{r=1}^{\infty} \sum_{k=r+1}^{\infty} \frac{1}{k^2(k-r)^2} = \sum_{k=2}^{\infty} \sum_{r=1}^{k-1} \frac{1}{k^2(k-r)^2} = \frac{\pi^4}{120} \quad (5.5)$$

$$\left(\sum_{k=1}^n f(k) \right) \left(\sum_{k=1}^n \frac{1}{f(k)} \right) = n + \sum_{r=1}^{n-1} \sum_{k=r+1}^n \frac{(f(k-r))^2 + (f(k))^2}{f(k-r)f(k)}, \quad n \geq 2 \quad (5.6)$$

$$\sum_{j=1}^n \frac{1}{j} = \frac{1}{n-1} \sum_{r=1}^{n-1} \sum_{k=1}^{n-r} \left(\frac{1}{k+r} + \frac{1}{k} \right), \quad n \geq 2 \quad (5.7)$$

5.2 Cauchy Convolution Formula

Remark 5.1 In this Section 5.2, we let $[x]$ denote the greatest integer in x .

$$\left(\sum_{k=0}^n f(k) \right)^2 = \sum_{k=0}^{2n} \sum_{i=[\frac{k}{n+1}](k-n)}^{k-[\frac{k+1}{n+1}](k-n)} f(i)f(k-i) \quad (5.8)$$

5.2.1 Applications of Cauchy Convolution Formula

Vandermonde Convolution

$$\sum_{i=0}^k \binom{n}{i} \binom{n}{k-i} = \binom{2n}{k} \quad (5.9)$$

$$\sum_{i=0}^k \binom{r}{i} \binom{q}{k-i} = \binom{r+q}{k}, \quad (5.10)$$

where k , r , and q are nonnegative integers.

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n} \quad (5.11)$$

$$\left(\sum_{i=0}^n \binom{n}{i} x^i f(i) \right) \left(\sum_{j=0}^n \binom{n}{j} x^j \varphi(j) \right) = \sum_{k=0}^{2n} x^k \sum_{i=0}^k \binom{n}{i} \binom{n}{k-i} f(i) \varphi(k-i) \quad (5.12)$$

$$\left(\sum_{k=0}^{\infty} a_k x^k \right)^2 = \sum_{k=0}^{\infty} \sum_{i=0}^k a_i a_{k-i} x^k \quad (5.13)$$

$$e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{i=0}^k \binom{k}{i} \quad (5.14)$$

$$(\cosh x)^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \sum_{j=0}^k \binom{2k}{2j} = \frac{1}{2} \cosh(2x) + \frac{1}{2} \quad (5.15)$$

$$\left(\sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} \right)^2 = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^4} x^k \quad (5.16)$$

Companion Binomial Theorem: Let n be a positive integer

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\text{infy}} \binom{k+n-1}{k} x^k = \left(\sum_{i=0}^{\infty} x^i \right)^n, \quad |x| < 1 \quad (5.17)$$

5.2.2 r^{th} Power of an Infinite Series

Let r be a positive integer. Assume $f(x) = \sum_{i=0}^{\infty} a_{1,i}x^i$. Then,

$$(f(x))^r = \sum_{k=0}^{\infty} a_{r,k}x^k, \quad (5.18)$$

where

$$a_{r,k} = \sum_{i=0}^k a_{r-1,i}a_{1,k-i}$$

5.3 Third Formula Convolution Formula

Remark 5.2 In Section 5.3, we let $[x]$ denote the greatest integer in x .

$$\sum_{i=0}^n a_i a_{n-i} = 2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} a_i a_{n-i} + \frac{1 + (-1)^n}{2} a_{\lfloor \frac{n}{2} \rfloor}^2, \quad n \geq 1 \quad (5.19)$$

Variation of Third Convolution Formula

$$\sum_{i=1}^n a_i a_{n-i+1} = 2 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i a_{n-i+1} + \frac{1 - (-1)^n}{2} a_{\lfloor \frac{n+1}{2} \rfloor}^2, \quad n \geq 2 \quad (5.20)$$

5.3.1 Applications of Third Convolution Formula

$$\sum_{i=1}^{2n} a_i a_{2n-i+1} = 2 \sum_{i=1}^n a_i a_{2n-i+1}, \quad n \geq 1 \quad (5.21)$$

$$\sum_{i=1}^{2n+1} a_i a_{2n-i+2} = 2 \sum_{i=1}^n a_i a_{2n-i+2} + a_{n+1}^2, \quad n \geq 1 \quad (5.22)$$

$$\sum_{k=1}^{n-1} a_k a_{2n-1-k} = \frac{a_{2n-1}}{2} \quad (5.23)$$

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{i}^2 = \frac{1}{2} \binom{2n}{n} - \frac{1 + (-1)^n}{4} \binom{n}{\lfloor \frac{n}{2} \rfloor}^2, \quad n \geq 1 \quad (5.24)$$

$$\sum_{i=0}^{n-1} \binom{2n}{i}^2 = \frac{1}{2} \binom{4n}{2n} - \frac{1}{2} \binom{2n}{n}^2, \quad n \geq 1 \quad (5.25)$$

$$\sum_{i=0}^n \binom{2n+1}{i}^2 = \frac{1}{2} \binom{4n+2}{2n+1}, \quad n \geq 1 \quad (5.26)$$

6 Finite Products: Elementary Properties

Remark 6.1 *In this chapter, we assume, unless otherwise specified, that a and p are nonnegative integers.*

6.1 Basic Properties

6.1.1 Commutativity Property

$$\prod_{k=a}^n f(k)\varphi(k) = \prod_{k=a}^n f(k) \prod_{k=a}^n \varphi(k) \quad (6.1)$$

Applications of Commutativity Property

$$\prod_{k=a}^n (f(k))^p = \left(\prod_{k=a}^n f(k) \right)^p, \quad p \geq 1 \quad (6.2)$$

$$\prod_{k=0}^n f(2k)f(2k+1) = \prod_{k=0}^{2n+1} f(k), \quad n \geq 0 \quad (6.3)$$

$$\prod_{k=0}^n \frac{f(2k)}{f(2k+1)} = \prod_{k=0}^{2n+1} (f(k))^{(-1)^k} \quad (6.4)$$

$$\prod_{k=0}^{\infty} \frac{f(2k)}{f(2k+1)} = \prod_{k=0}^{\infty} (f(k))^{(-1)^k} \quad (6.5)$$

6.1.2 Exponent Property

$$\prod_{k=a}^n \alpha^{f(k)} = \alpha^{\sum_{k=a}^n f(k)} \quad (6.6)$$

Applications of Exponent Property

$$\prod_{k=a}^n \alpha = \alpha^{n-a+1} \quad (6.7)$$

$$\prod_{k=a}^n \alpha^k = \alpha^{\frac{n^2 - a^2 + a}{2}} \quad (6.8)$$

$$\prod_{k=0}^n x^{(-1)^k \binom{n}{k}} = 1, \quad n \neq 0, \quad x \neq 0 \quad (6.9)$$

otherwise, the previous product equals x , when $n = 0$.

$$\prod_{k=0}^n x^{\binom{n}{k}} = x^{2^n} \quad (6.10)$$

6.1.3 Logarithm of Product Property

$$\log_b \prod_{k=a}^n (f(k))^p = p \sum_{k=a}^n \log_b f(k) \quad (6.11)$$

6.1.4 Product as an Exponential Function

$$\prod_{k=a}^n f(k) = e^{\sum_{k=a}^n \ln f(k)} \quad (6.12)$$

$$\prod_{k=a}^n (1 + f(k)) = e^{\sum_{k=a}^n f(k)} e^{\sum_{k=a}^n \sum_{j=2}^{\infty} (-1)^{j-1} \frac{(f(k))^j}{j}}, \quad |f(x)| < 1, \quad a \leq x \leq n \quad (6.13)$$

6.1.5 Factorial as a Finite Product

$$\prod_{k=1}^n k = n! \quad (6.14)$$

Remark 6.2 *In the following identity, we assume b is a nonnegative integer. If the reader wants to let b be an arbitrary complex number, then he or she must use the convention $\Gamma(b) = (b-1)!$.*

$$\prod_{k=a}^n (k+b) = \frac{(n+b)!}{(b+a-1)!} \quad (6.15)$$

Remark 6.3 *In the following identity, we assume b is a positive integer greater than n . Otherwise, the reader must use the fact that $\Gamma(b) = (b-1)!$ whenever b is a complex number which is not a negative integer.*

$$\prod_{k=a}^n (b-k) = \frac{(b-a)!}{(b-n-1)!} \quad (6.16)$$

Remark 6.4 *In the next eight identities, x is any complex number for which the corresponding factorial expression will be defined via the Gamma function (see Remark 6.2).*

$$\prod_{j=0}^n (2j+x) = 2^{n+1} \frac{(n+\frac{x}{2})!}{(\frac{x}{2}-1)!}, \quad n \geq 0 \quad (6.17)$$

$$\prod_{j=0}^n (2j-x) = 2^{n+1} \frac{(n-\frac{x}{2})!}{(-\frac{x}{2}-1)!}, \quad n \geq 0 \quad (6.18)$$

$$\prod_{j=0}^n (4j^2 - x^2) = 2^{2n+2} \frac{(n+\frac{x}{2})! (n-\frac{x}{2})!}{(\frac{x}{2}-1)! (-\frac{x}{2}-1)!}, \quad n \geq 0 \quad (6.19)$$

$$\prod_{j=0}^n (2j+1+x) = 2^{n+1} \frac{(n+\frac{x+1}{2})!}{(\frac{x+1}{2}-1)!}, \quad n \geq 0 \quad (6.20)$$

$$\prod_{j=0}^n (2j+1-x) = 2^{n+1} \frac{(n+\frac{-x+1}{2})!}{(\frac{-x+1}{2}-1)!}, \quad n \geq 0 \quad (6.21)$$

$$\prod_{j=0}^n ((2j+1)^2 - x^2) = 2^{2n+2} \binom{n + \frac{x+1}{2}}{n+1} \binom{n - \frac{x-1}{2}}{n+1} ((n+1)!)^2, \quad n \geq 0 \quad (6.22)$$

$$\prod_{j=0}^n (x^2 - j^2) = \frac{x(x+n)!}{(x-n-1)!}, \quad n \geq 0 \quad (6.23)$$

$$\prod_{j=0}^n (j^2 - x^2) = (-1)^{n+1} \frac{x(x+n)!}{(x-n-1)!}, \quad n \geq 0 \quad (6.24)$$

Remark 6.5 In the following identity, we assume b is a positive integer. The resulting factorial expressions are evaluated by use of the Gamma function (see Remark 6.2).

$$\prod_{k=1}^n (1 + bk) = \frac{b^n (n + \frac{1}{b})!}{(\frac{1}{b})!} \quad (6.25)$$

$$\prod_{k=1}^n (2k+1) = \frac{(2n+1)!}{2^n n!}, \quad n \geq 1 \quad (6.26)$$

$$\prod_{k=1}^n 2k = 2^n n!, \quad n \geq 1 \quad (6.27)$$

$$\prod_{k=1}^n (1+k)^p = ((n+1)!)^p \quad (6.28)$$

6.1.6 Binomial Coefficient as Finite Product

$$\prod_{k=1}^n \left(1 + \frac{a}{k}\right) = \prod_{k=1}^a \left(1 + \frac{n}{k}\right) = \binom{n+a}{a}, \quad n \geq 1, \quad a \geq 1 \quad (6.29)$$

$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^p = n^p, \quad n \geq 1 \quad (6.30)$$

$$\prod_{k=1}^n \left(1 + \frac{n}{k}\right) = \frac{(2n)!}{(n!)^2}, \quad n \geq 1 \quad (6.31)$$

$$\prod_{k=1}^{n-1} \left(1 - \frac{n}{k}\right) = (-1)^{n+1}, \quad n \geq 2 \quad (6.32)$$

Remark 6.6 In the following three identities, we assume a is a positive integer. The corresponding factorials are evaluated via the Gamma function (see Remark 6.2).

$$\prod_{k=1}^n \left(1 \pm \frac{1}{ak}\right) = \binom{n \pm \frac{1}{a}}{n}, \quad n \geq 1 \quad (6.33)$$

$$\prod_{k=1}^n \left(1 - \frac{1}{a^2 k^2}\right) = \frac{(n + \frac{1}{a})! (n - \frac{1}{a})!}{(\frac{1}{a})! (\frac{-1}{a})! (n!)^2}, \quad n \geq 1 \quad (6.34)$$

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{a^2 k^2}\right) = \frac{1}{(\frac{1}{a})! (\frac{-1}{a})!} = \frac{\sin(\frac{\pi}{a})}{\frac{\pi}{a}} \quad (6.35)$$

$$\prod_{j=1}^{k-1} \left(1 - \frac{n^2}{j^2}\right) = \frac{(-1)^{k-1} k^2}{2^{2k} n^2} \binom{2k}{2j} \sum_{j=0}^k \binom{2n}{2j} \binom{n-j}{k-j} \quad (6.36)$$

$$\prod_{j=1}^{n-1} \left(1 - \frac{n^2}{j^2}\right) = \frac{(-1)^{n-1} (2n)}{2} \binom{2n}{n}, \quad n \geq 1 \quad (6.37)$$

6.1.7 Index Shift Formula

$$\prod_{k=a}^n f(k) = \prod_{k=a}^n f(n - k + a) \quad (6.38)$$

Applications of Index Shift Formula

$$\prod_{k=1}^n k^2 = (n!)^2, \quad n \geq 1 \quad (6.39)$$

$$\prod_{k=1}^{n-1} (kn - k^2) = ((n-1)!)^2, \quad n \geq 2 \quad (6.40)$$

$$\prod_{k=1}^n \left(\frac{n+1}{k} - 1\right) = 1, \quad n \geq 1 \quad (6.41)$$

$$\prod_{k=0}^n k! = \prod_{k=0}^{n-1} (k+1)^{n-k} = \prod_{k=0}^{n-1} (n-k)^{k+1} = \prod_{k=0}^n (n-k)! \quad (6.42)$$

$$\prod_{k=1}^n \frac{1}{k!} = \prod_{k=0}^{n-1} n^{-(k+1)} \prod_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^{-(k+1)}, \quad n \geq 1 \quad (6.43)$$

6.1.8 Two Cancellation Properties

$$\prod_{k=a}^n \frac{f(k+1)}{f(k)} = \frac{f(n+1)}{f(a)} \quad (6.44)$$

$$\prod_{k=a}^n \frac{f(k-1)}{f(k)} = \frac{f(a-1)}{f(n)}, \quad a \geq 1 \quad (6.45)$$

Applications of the Cancellation Properties

$$\prod_{k=1}^n \left(1 + \frac{1}{k}\right)^p = (n+1)^p, \quad n \geq 1 \quad (6.46)$$

$$\prod_{r=1}^n \prod_{k=1}^r \left(1 + \frac{1}{k}\right)^p = \prod_{r=1}^n (r+1)^p = ((n+1)!)^p, \quad n \geq 1 \quad (6.47)$$

$$\prod_{k=a}^n \left(1 + \frac{1}{k}\right)^p = \left(\frac{n+1}{a}\right)^p, \quad a \geq 1 \quad (6.48)$$

$$\prod_{k=a}^n \left(1 - \frac{1}{k}\right)^p = \left(\frac{a-1}{n}\right)^p, \quad a \geq 2 \quad (6.49)$$

$$\prod_{k=a}^n \left(1 - \frac{1}{k^2}\right)^p = \left(\frac{a-1}{a} \left(1 + \frac{1}{n}\right)\right)^p, \quad a \geq 2 \quad (6.50)$$

$$\prod_{k=a}^{\infty} \left(1 - \frac{1}{k^2}\right)^p = \left(\frac{a-1}{a}\right)^p, \quad a \geq 2 \quad (6.51)$$

$$\prod_{k=2}^n \frac{k+1}{k-1} = \frac{n^2+n}{2} = \sum_{k=1}^n k, \quad n \geq 2 \quad (6.52)$$

$$\sum_{k=2}^n \ln(k+1) - \sum_{k=2}^n \ln(k-2) = \ln \sum_{k=1}^n k = \ln \left(\frac{n^2+n}{2} \right), \quad n \geq 2 \quad (6.53)$$

$$\prod_{k=1}^n \frac{(1+k)^{2p}}{k^p} = (n+1)^{2p} (n!)^p, \quad n \geq 1 \quad (6.54)$$

$$\prod_{k=1}^n \left(\frac{r+k}{k} \right)^k = \frac{1}{(n!)^r} \prod_{j=1}^r (n+j)^{n-r+j}, \quad r, n \geq 1 \quad (6.55)$$

Remark 6.7 In the following identity, we assume r is a positive integer such that $r \geq n + 1$, for fixed integer $n \geq 1$. If the reader prefers to let r represent an arbitrary complex number, the factorials must be evaluated by the Gamma function (See Remark 6.2).

$$\prod_{k=1}^n \frac{r-k}{k+1} = \frac{\binom{r}{n+1}}{\binom{r}{1}} = \frac{(r-1)!}{(n+1)!(r-n-1)!} \quad (6.56)$$

Remark 6.8 In the following identity, we assume $r = 0$ or $r = 1$. If the reader prefers to let r be any complex number which is not a positive integer greater than or equal to 2, he or she should ignore the binomial coefficient representation and evaluate the factorial by the Gamma function (See Remark 6.2).

$$\prod_{k=1}^n \frac{k+1}{k+1-r} = \frac{\binom{n+1}{r}}{\binom{n+1}{1}} = \frac{(n+1)!(1-r)!}{(n+1-r)!}, \quad n \geq 1 \quad (6.57)$$

6.1.9 Three Product Identities From Identity (3.7)

$$\prod_{k=a}^n \left(1 + \frac{1}{k}\right)^k = \frac{(a-1)!(n+1)^n}{a^{a-1}n!}, \quad a \neq 1 \quad (6.58)$$

$$\prod_{k=1}^n k^{2k-1} = (n+1)^{n^2} \prod_{k=1}^n \left(\frac{k}{k+1}\right)^{k^2}, \quad n \geq 1 \quad (6.59)$$

$$\prod_{k=1}^n k^{2k-1} \prod_{k=1}^n \left(1 + \frac{1}{k}\right)^{k^2} = (n+1)^{n^2}, \quad n \geq 1 \quad (6.60)$$

6.1.10 Iterative Product Formula

$$\prod_{i=0}^n \prod_{k=0}^i f(k) = \prod_{k=0}^n (f(k))^{n-k+1} = \prod_{k=0}^n (f(n-k))^{k+1} \quad (6.61)$$

Applications of Iterative Product Formula

$$\prod_{k=1}^n k^k k! = (n!)^{n+1}, \quad n \geq 1 \quad (6.62)$$

$$\prod_{k=1}^n k^{k-1} k! = (n!)^n, \quad n \geq 1 \quad (6.63)$$

6.2 Trigonometric Products

Remark 6.9 In Section 6.2, we assume the reader is familiar with the Weirstrass Factorization Theorem. The reader may find this important theorem on the Wikipedia website.

Product for $\sin(\theta)$

$$\sin(\theta) = \theta \prod_{k=1}^{\infty} \left(1 - \frac{\theta^2}{k^2\pi^2}\right) \quad (6.64)$$

Product for $\cos(\theta)$

$$\cos(\theta) = \prod_{k=1}^{\infty} \left(1 - \frac{4\theta^2}{(2k-1)^2\pi^2}\right) \quad (6.65)$$

7 Intermediate Level Calculations Involving Products

Remark 7.1 In the following chapter, we assume, unless otherwise specified, that r and n are positive integers.

7.1 Defining $n!$ as a Product Limit

$$\lim_{r \rightarrow \infty} \frac{r^n n!}{(r+1)(r+2)\dots(n+r)} = n! \quad (7.1)$$

$$\lim_{r \rightarrow \infty} \frac{r^n r!}{(n+1)(n+2)\dots(n+r)} = n! \quad (7.2)$$

$$\lim_{r \rightarrow \infty} r^n \prod_{k=1}^r \frac{1}{\left(1 + \frac{n}{k}\right)} = n! \quad (7.3)$$

$$\lim_{r \rightarrow \infty} r^{-n} \prod_{k=1}^r \left(1 + \frac{n}{k}\right) = \frac{1}{n!} \quad (7.4)$$

7.2 Products From a Recursive Sequence

Remark 7.2 In Section 7.2, we assume a such that $a \geq 2$. We assume n is a nonnegative integer. We define the sequence $(u_{0,n})_{n=0}^{\infty}$ by the recursive definition $(an - 1)u_{0,n} = u_{0,n+1}$ with $u_{0,1} = 1$.

$$u_{0,n+1} = u_{0,n-r+1} \prod_{k=0}^{r-1} (an - ka - 1) \quad (7.5)$$

$$u_{0,n+1} = u_{0,0} \prod_{k=0}^n (an - ka - 1) \quad (7.6)$$

Remark 7.3 In the following four identities, we let $a = 2$. Also, any noninteger factorial is evaluated by the Gamma function, i.e. $\Gamma(x + 1) = x!$, for all complex numbers x , except negative integers.

$$u_{0,n} = \frac{2^{n-1}}{\sqrt{\pi}} \Gamma\left(\frac{2n-1}{2}\right) \quad (7.7)$$

$$\prod_{k=0}^n (2n - 2k - 1) = -\frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)! \quad (7.8)$$

$$\prod_{k=0}^{r-1} (2n - 2k - 1) = 2^r \frac{\left(n - \frac{1}{2}\right)!}{\left(n - r - \frac{1}{2}\right)!} \quad (7.9)$$

$$\prod_{k=0}^n (2k + 1) = \frac{2^{n+1} \sqrt{\pi}}{(-1)^{n+1} \left(-n - \frac{3}{2}\right)!} \quad (7.10)$$

7.3 Applications of Binomial Coefficient as Product Formula

$$\binom{n + \frac{2r+1}{2}}{n} = \frac{1}{2^{2n}} \binom{2n}{2} \prod_{k=0}^r \frac{2n + 2k + 1}{2k + 1}, \quad (7.11)$$

where n is a positive integer and r is a nonnegative integer.

$$\binom{n + \frac{1}{2}}{n} = \binom{2n}{n} \frac{2n + 1}{2^{2n}} \quad (7.12)$$

$$\binom{n + \frac{3}{2}}{n} = \binom{2n}{n} \frac{(2n + 1)(2n + 3)}{3 * 2^{2n}} \quad (7.13)$$

$$\binom{n + \frac{5}{2}}{n} = \binom{2n}{n} \frac{(2n + 1)(2n + 3)(2n + 5)}{3 * 5 * 2^{2n}} \quad (7.14)$$

$$\binom{n + \frac{7}{2}}{n} = \binom{2n}{n} \frac{(2n + 1)(2n + 3)(2n + 5)(2n + 7)}{3 * 5 * 7 * 2^{2n}} \quad (7.15)$$

Remark 7.4 In the following three identities, any non integer factorials are evaluated via the Gamma function (see Remark (7.3))

$$\binom{n + \frac{\alpha}{2}}{n} = \frac{(2n + \alpha)! \left(\frac{\alpha-1}{2}\right)!}{2^{2n} \alpha! n! \left(n + \frac{\alpha-1}{2}\right)!}, \quad (7.16)$$

where n is a positive integer and α is a nonnegative integer.

$$\binom{n + k}{n} = \frac{(2n + 2k)! \left(\frac{2k-1}{2}\right)!}{2^{2n} (2k)! n! \left(n + \frac{2k-1}{2}\right)!}, \quad (7.17)$$

where n is a positive integer and k is any real number.

$$\binom{2n + 2k}{2n} \binom{2n}{n} = 2^{2n} \binom{n + k}{n} \binom{n + \frac{2k-1}{2}}{n}, \quad (7.18)$$

where n is a nonnegative integer and k is any real number.

Remark 7.5 In the following three identities x is an arbitrary complex number, h is any nonzero complex number and n is a positive integer.

$$\prod_{k=0}^{n-1} (x + kh) = h^n n! \binom{\frac{x}{h} + n - 1}{n} \quad (7.19)$$

$$\prod_{k=0}^{n-1} (x - kh) = h^n n! \binom{\frac{x}{h}}{n} \quad (7.20)$$

$$\lim_{h \rightarrow 0} h^n n! \binom{\frac{x}{h}}{n} = x^n \quad (7.21)$$

7.4 Induction on Three Product Expansions

Remark 7.6 In Section 7.4, we let $[x]$ denote the greatest integer in x .

7.4.1 First Product Expansion

$$\prod_{k=2}^n k^{2k-1} = \prod_{k=1}^{n^2-1} (1 + [\sqrt{k}]), \quad n \geq 2 \quad (7.22)$$

$$([\sqrt{n}] + 1)^{n - [\sqrt{n}]^2 + 1} \prod_{k=2}^{[\sqrt{n}]} k^{2k-1} = \prod_{k=1}^n (1 + [\sqrt{k}]), \quad n \geq 2 \quad (7.23)$$

7.4.2 Second Product Expansion

$$\prod_{k=1}^n \left(1 + \frac{(-1)^{k-1}}{k} \right) = 1 + \frac{1 - (-1)^n}{2n}, \quad n \geq 1 \quad (7.24)$$

$$\prod_{k=2}^n \left(1 - \frac{(-1)^k}{k} \right) = \frac{1}{2} + \frac{1 - (-1)^n}{4n}, \quad n \geq 2 \quad (7.25)$$

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k-1}}{k} \right) = 1 \quad (7.26)$$

7.4.3 Third Product Expansion

$$\prod_{k=2}^n \left(1 + \frac{(-1)^k}{k}\right) = 1 + \frac{1 + (-1)^n}{2n}, \quad n \geq 2 \quad (7.27)$$

7.5 Three Product Functions

7.5.1 First Product Function

$$\prod_{i=1}^n (1 + x^i)(1 - x^{2i-1}) = \prod_{j=n+1}^{2n} (1 - x^j) = \prod_{j=1}^n (1 - x^{j+n}) \quad (7.28)$$

7.5.2 Second Product Function

$$\prod_{i=1}^n (1 - x^{2i})(1 - x^{2i-1}) = \prod_{j=1}^{2n} (1 - x^j) \quad (7.29)$$

7.5.3 Third Product Function

$$\prod_{i=1}^n (1 + x^{2i})(1 + x^{2i-1}) = \prod_{j=1}^n (1 + x^j) \quad (7.30)$$

7.6 The Product Functions $\prod_{k=0}^n x^{a^k}$ and $\prod_{k=0}^n (1 + x^{a^k})$

Remark 7.7 In Section 7.6, we assume a is any nonzero real number, except 1. Also, we may assume that x is any nonzero complex number for which the products and resulting functions are defined.

$$\prod_{k=0}^n x^{a^k} = x^{\frac{a^{n+1}-1}{a-1}} \quad (7.31)$$

$$\prod_{k=0}^n \left(1 + \frac{1}{x^{2^k}}\right) = \frac{x}{x-1} \left(1 - \frac{1}{x^{2^{n+1}}}\right) \quad (7.32)$$

$$\prod_{k=0}^n (1 + x^{2^k}) = \frac{x^{2^{n+1}} - 1}{x - 1} \quad (7.33)$$

$$\prod_{k=0}^{n-1} (1 + x^{2^k}) = \frac{1 - x^{2^n}}{1 - x} \quad (7.34)$$

$$\sum_{k=1}^n x^{2^k} (x^{2^k} - 1) = x^2 - x^{2^{n+1}}, \quad n \geq 1 \quad (7.35)$$

$$\sum_{k=1}^n \frac{1}{x^{2^k}} \left(1 - \frac{1}{x^{2^k}}\right) = \frac{1}{x^2} - \frac{1}{x^{2^{n+1}}}, \quad n \geq 1 \quad (7.36)$$

$$\sum_{k=1}^{\infty} \frac{1}{x^{2^k}} \left(1 - \frac{1}{x^{2^k}}\right) = \frac{1}{x^2}, \quad |x| \geq 1 \quad (7.37)$$

Remark 7.8 In the following four identities, we let $x = 2$.

$$\prod_{k=0}^n 2^{2^k} = 2^{2^{n+1} - 1} \quad (7.38)$$

$$\prod_{k=0}^n \left(1 + \frac{1}{2^{2^k}}\right) = 2 \left(1 - \frac{1}{2^{2^{n+1}}}\right) \quad (7.39)$$

$$\prod_{k=0}^n (1 + 2^{2^k}) = 2^{2^{n+1}} - 1 \quad (7.40)$$

$$\prod_{k=0}^{\infty} \left(1 + \frac{1}{2^{2^k}}\right) = 2 \quad (7.41)$$

7.6.1 Product Identities Involving Geometric Series

$$\prod_{k=1}^n \sum_{i=0}^{r-1} x^{ir^{k-1}} = \sum_{j=0}^{r^n-1} x^j = \frac{1 - x^{r^n}}{1 - x}, \quad n, r \geq 1, x \neq 1 \quad (7.42)$$

$$\prod_{k=1}^n (1 + x^{3^{k-1}} + x^{2 \cdot 3^{k-1}}) = \frac{1 - x^{3^n}}{1 - x}, \quad n \geq 1, x \neq 1 \quad (7.43)$$

8 Relationships Between Finite Series and Finite Products

8.1 Series as a Product

$$\sum_{k=1}^n f(k) = f(1) \prod_{k=2}^n \left(1 + \frac{f(k)}{\sum_{i=1}^{k-1} f(i)} \right), \quad n \geq 2 \quad (8.1)$$

8.1.1 Applications of Series as Product Formula

$$\sum_{k=1}^n \frac{1}{2^{k-1}} = \prod_{k=2}^n \left(1 + \frac{1}{2^k - 2} \right) = 2 - \frac{1}{2^{n-1}} \quad (8.2)$$

$$\prod_{k=2}^{\infty} \left(1 + \frac{1}{2^k - 2} \right) = 2 \quad (8.3)$$

$$2 \sum_{k=1}^n \frac{1}{k(k+1)} = \prod_{k=2}^n \left(1 + \frac{1}{k^2 - 1} \right) = \frac{2n}{n+1} \quad (8.4)$$

$$\prod_{k=2}^{\infty} \left(1 + \frac{1}{k^2 - 1} \right) = 2 \quad (8.5)$$

$$\sum_{k=1}^n \frac{1}{k(k+2)} = \frac{1}{3} \prod_{k=2}^n \left(1 + \frac{4(k+1)}{(k-1)(k+2)(3k+2)} \right) = \frac{1}{4} \frac{n(3n+5)}{(n+1)(n+2)} \quad (8.6)$$

$$\prod_{k=2}^{\infty} \left(1 + \frac{4(k+1)}{(k-1)(k+2)(3k+2)} \right) = \frac{9}{4} \quad (8.7)$$

$$2 \sum_{k=1}^n \frac{k}{(k+1)!} = \prod_{k=2}^n \left(1 + \frac{k}{(k+1)! - k - 1} \right) = 2 - \frac{2}{(n+1)!} \quad (8.8)$$

$$\prod_{k=2}^{\infty} \left(1 + \frac{k}{(k+1)! - k - 1} \right) = 2 \quad (8.9)$$

$$\sum_{k=1}^n \frac{k(k+2)}{(k+1)^2(k!)^2} = \frac{3}{4} \prod_{k=2}^n \left(1 + \frac{k(k+2)}{(k+1)^2((k!)^2 - 1)} \right) = 1 - \frac{1}{((n+1)!)^2} \quad (8.10)$$

$$\prod_{k=2}^{\infty} \left(1 + \frac{k(k+2)}{(k+1)^2((k!)^2 - 1)} \right) = \frac{4}{3} \quad (8.11)$$

8.2 Product as a Series

$$\prod_{k=1}^n (1 + f(k)) = 1 + f(1) + \sum_{k=2}^n f(k) \prod_{i=1}^{k-1} (1 + f(i)) \quad (8.12)$$

8.2.1 Applications of Product as Series Formula

Remark 8.1 *In the following five identities, we assume r and n are positive integers.*

$$\sum_{k=1}^n \binom{k+r-1}{r} \frac{1}{k} = \frac{1}{r} \binom{n+r}{r} - \frac{1}{r} \quad (8.13)$$

$$\lim_{r \rightarrow \infty} \frac{1}{r^{n-1}} \sum_{k=1}^n \binom{k+r-1}{r} \frac{1}{k} = \frac{1}{n!} \quad (8.14)$$

$$\sum_{k=1}^n \binom{k+r}{k} \frac{1}{k+r} = \frac{\binom{n+r}{r} - 1}{r} \quad (8.15)$$

$$\lim_{r \rightarrow \infty} \frac{1}{r^{n-1}} \sum_{k=1}^n \binom{k+r}{r} \frac{1}{k+r} = \frac{1}{n!} \quad (8.16)$$

$$\lim_{r \rightarrow 0} \frac{\binom{n+r}{r} - 1}{r} = \sum_{k=1}^n \frac{1}{k} \quad (8.17)$$

Remark 8.2 *In the following two identities, we assume $0^0 = 1$.*

$$\sum_{k=0}^n \frac{(1+k)^k - k^k}{k!} = \frac{(n+1)^n}{n!} \quad (8.18)$$

$$\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{(1+k)^k - k^k}{k!} \right)^{\frac{1}{n}} = e \quad (8.19)$$

8.3 Schlomilch Series to Product Identity

$$\sum_{i=0}^n u_i = \frac{u_0}{1} \prod_{k=0}^{n-1} \frac{\sum_{i=0}^{k+1} u_i}{\sum_{i=0}^k u_i} \quad (8.20)$$

8.4 Schlomilch Product to Series Identity

$$\prod_{j=0}^n v_j = v_0 + v_0(v_1 - 1) + v_0v_1(v_2 - 1) + \dots + v_0v_1v_2\dots v_{n-1}(v_n - 1) \quad (8.21)$$

Fundamentals of Series: Table II: Examples of Series Which Appear in Calculus

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 The Binomial Theorem

Remark 1.1 *In this table, unless otherwise specified, n and r are nonnegative integers, and x and z are arbitrary real or complex numbers. We also assume that for any real number x , $[x]$ is the greatest integer in x .*

1.1 Binomial Theorem

1.1.1 Basic Form with Integer Power

$$(x + a)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} a^k, \text{ where } a \text{ is an arbitrary real or complex number} \quad (1.1)$$

1.1.2 Newton's Binomial Theorem

$$(1 + x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k, \text{ where } z \text{ is a real or complex number and } |x| < 1 \quad (1.2)$$

1.1.3 Applications of Newton's Binomial Theorem

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{2^{2k}} = \frac{1}{\sqrt{1-x}}, \quad |x| < 1 \quad (1.3)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{3k}} = \sqrt{2} \quad (1.4)$$

$$\sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} \frac{1}{2^{2k}} = \frac{\sqrt{2}}{2} \quad (1.5)$$

$$-\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{2^{2k}(2k-1)} = \sqrt{1-x}, \quad |x| < 1 \quad (1.6)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}(2k-1)} = 0 \quad (1.7)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{2^{2k}(2k-1)} = -\sqrt{2} \quad (1.8)$$

1.2 Companion Binomial Theorem

Remark 1.2 *In Section 1.2, we assume p is a nonnegative integer. We also assume a and b are arbitrary real or complex numbers.*

Companion Binomial Theorem

$$\sum_{n=0}^{\infty} \binom{n+p}{n} x^n = \sum_{n=p}^{\infty} \binom{n}{p} x^{n-p} = \frac{1}{(1-x)^{p+1}}, \quad |x| < 1 \quad (1.9)$$

1.2.1 Applications of Companion Binomial Theorem

$$\frac{1}{(a+b)^{p+1}} = \frac{1}{a^{p+1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+p}{k} \frac{b^k}{a^k}, \quad \left| \frac{b}{a} \right| < 1 \quad (1.10)$$

$$\frac{1}{(a-b)^{p+1}} = \frac{1}{a^{p+1}} \sum_{k=0}^{\infty} \binom{k+p}{k} \frac{b^k}{a^k}, \quad \left| \frac{b}{a} \right| < 1 \quad (1.11)$$

$$\frac{1}{(1+x^m)^{p+1}} = \sum_{k=0}^{\infty} (-1)^k \binom{k+p}{k} x^{mk}, \quad |x| < 1, \quad m \in \Re \quad (1.12)$$

$$\frac{1}{(1-x^m)^{p+1}} = \sum_{k=0}^{\infty} \binom{k+p}{k} x^{mk}, \quad |x| < 1, m \in \mathbb{R} \quad (1.13)$$

$$\frac{1}{(x^m + 1)^{p+1}} = \sum_{k=0}^{\infty} (-1)^k \binom{k+p}{k} x^{-m(k+p+1)}, \quad |x| > 1, m \in \mathbb{R} \quad (1.14)$$

$$\frac{1}{(x^m - 1)^{p+1}} = \sum_{k=0}^{\infty} \binom{k+p}{k} x^{-m(k+p+1)}, \quad |x| > 1, m \in \mathbb{R} \quad (1.15)$$

1.2.2 Finite Version of Companion Binomial Theorem

$$\sum_{k=0}^n \binom{n+k}{k} \frac{1}{2^k} = 2^n \quad (1.16)$$

Variation of Finite Companion Binomial Theorem

$$\sum_{k=0}^n \binom{2n-k}{n} 2^k = 2^{2n} \quad (1.17)$$

Application of Finite Companion Binomial Theorem

$$\sum_{k=1}^{\infty} \binom{2n+k}{n} \frac{1}{2^k} = 2^{2n} \quad (1.18)$$

1.3 Binomial Theorem with Complex Exponents

Remark 1.3 *The material in Section 1.3 is found in T. J. I'a. Bromwich's Introduction to the Theory of Infinite Series, Second Edition, 1949, Chapter 9, Article 96.*

Let $\alpha, \beta \in \mathfrak{R}$. Let $i^2 \equiv -1$. Then,

$$(1 + x)^{\alpha + \beta i} = \sum_{k=0}^{\infty} \binom{\alpha + \beta i}{k} x^k, \quad (1.19)$$

where

- a. *The series is absolutely convergent for $|x| < 1$.*
- b. *If $\alpha > 0$, the series converges absolutely on the circle $|x| = 1$. Hence, the series is uniformly convergent within and on the circle $|x| = 1$.*
- c. *If $-1 < \alpha \leq 0$, the series converges on the circle $|x| = 1$ except at $x = -1$.*
- d. *If $\alpha \leq -1$, the series diverges everywhere on the circle $|x| = 1$.*

1.4 Applications of the Binomial Theorem

1.4.1 Derivatives of the Binomial Series

$$\sum_{k=0}^n \binom{n}{k} \binom{k}{r} x^k = x^r (1 + x)^{n-r} \binom{n}{r} \quad (1.20)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{k}{r} = 2^{n-r} \binom{n}{r} \quad (1.21)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{k}{r} 2^k = 2^r 3^{n-r} \binom{n}{r} \quad (1.22)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{r} x^k = (-1)^r x^r (1 - x)^{n-r} \binom{n}{r} \quad (1.23)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k 2^k = 2n(-1)^n \quad (1.24)$$

1.4.2 Expansions of $(1 - x)^{-\frac{1}{2}}$

$$\sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} x^k = \frac{1}{\sqrt{1-x}}, \quad |x| < 1 \quad (1.25)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{(1+x)^{2k+1}} = \frac{1}{1-x}, \quad |x| < 1 \text{ or } \left| \frac{x}{1+x} \right| < 1 \quad (1.26)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \binom{k}{r} \frac{x^k}{2^{2k}} = (1-x)^{-\frac{2r+1}{2}} \left(\frac{x}{4}\right)^r \binom{2r}{r}, \quad |x| < 1 \quad (1.27)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \binom{k}{r} \frac{1}{2^{3k}} = \frac{\sqrt{2}}{2^{2r}} \binom{2r}{r} \quad (1.28)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{k}{2^{3k}} = \frac{\sqrt{2}}{2} \quad (1.29)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{k}{2^{2k}} x^k = \frac{x}{2(1-x)^{\frac{3}{2}}}, \quad |x| < 1 \quad (1.30)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{2k+1}{2^{2k}} x^k = \frac{1}{(1-x)^{\frac{3}{2}}}, \quad |x| < 1 \quad (1.31)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{2k+1}{2^{3k}} = \sqrt{8} \quad (1.32)$$

Bruckman's Formula Version 1

$$\sum_{k=0}^n \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{n-k} \frac{1}{(2k+1)(2n-2k+1)} = \frac{-1}{2(n+1)^2 \binom{-\frac{1}{2}}{n+1}} \quad (1.33)$$

Bruckman's Formula Version 2

$$\begin{aligned} \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{(2k+1)(2n-2k+1)} &= \frac{2^{4n+1}}{(n+1)^2 \binom{2n+2}{n+1}} \\ &= \frac{2^{4n}}{(n+1)(2n+1) \binom{2n}{n}} \end{aligned} \quad (1.34)$$

1.4.3 Expansions of $(1 - x)^{\frac{1}{2}}$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{4^k(2k-1)} = -(1-x)^{\frac{1}{2}}, \quad |x| < 1 \quad (1.35)$$

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{z^{k+1}}{2k+1} = \frac{1 - (1-4z)^{\frac{1}{2}}}{2}, \quad |z| < \frac{1}{4} \quad (1.36)$$

1.4.4 Evaluation of $\sum_{k=0}^n \binom{n}{k} \frac{x^k}{k+1}$

$$\sum_{k=0}^n \binom{n}{k} \frac{x^k}{k+1} = \frac{(x+1)^{n+1} - 1}{(n+1)x}, \quad x \neq 0 \quad (1.37)$$

$$\sum_{k=0}^n \binom{n}{k} \frac{1}{k+1} = \frac{2^{n+1} - 1}{n+1} \quad (1.38)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+1} = \frac{1}{n+1} \quad (1.39)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{2^k}{k+1} = \frac{1}{2n+1} \quad (1.40)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n+1}{k} \frac{2^k}{k+1} = \frac{2^{2n}}{n+1} \quad (1.41)$$

$$\sum_{k=1}^{2n} (-1)^k \binom{2n}{k-1} \frac{2^k}{k+1} = \frac{2^{2n}}{n+1} - \frac{1}{2n+1}, \quad n \geq 1 \quad (1.42)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{x^{2k}}{2k+1} = \frac{(x+1)^{n+1} - (1-x)^{n+1}}{2(n+1)x}, \quad x \neq 0 \quad (1.43)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{1}{2k+1} = \frac{2^n}{n+1} \quad (1.44)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{x^{2k}}{k+1} = \frac{(x+1)^{n+1} + (1-x)^{n+1} - 2}{(n+1)x^2}, \quad x \neq 0 \quad (1.45)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{1}{k+1} = \frac{2^{n+1} - 2}{n+1} \quad (1.46)$$

$$\sum_{k=0}^n \binom{n}{k} \frac{k!}{(k+r)!} x^k = \frac{n! \left((x+1)^{n+r} - \sum_{k=0}^{r-1} \binom{n+r}{k} x^k \right)}{(n+r)! x^r}, \quad x \neq 0, r \geq 1 \quad (1.47)$$

$$\sum_{k=0}^n \binom{n}{k} \frac{x^{k+r}}{\binom{k+r}{k}} = \frac{(x+1)^{n+r} - \sum_{i=0}^{r-1} \binom{n+r}{i} x^i}{\binom{n+r}{n}}, \quad r \geq 1 \quad (1.48)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k!}{(k+r)!} = (-1)^{r+1} n! \sum_{k=0}^{r-1} (-1)^k \frac{1}{(n+r-k)! k!}, \quad r \geq 1 \quad (1.49)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{k+r}{k}} \left((1-x)^{k+r} - \sum_{\alpha=0}^{r-1} (-1)^\alpha \binom{k+r}{\alpha} x^\alpha \right) = (-1)^r \frac{x^{n+r}}{\binom{n+r}{n}}, \quad r \geq 1 \quad (1.50)$$

1.4.5 Expansions of $(t-a)^{n-1}(t-(a+nx))$

Remark 1.4 *In the identities related to the expansion of $(t-a)^{n-1}(t-(a+nx))$, we assume, unless otherwise specified, that x , t , and a are nonzero real or complex numbers.*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} t^{n-k} a^{k-1} (a+kx) = (t-a)^{n-1} (t-(a+nx)), \quad n \geq 1 \quad (1.51)$$

$$\lim_{a \rightarrow 0} \sum_{k=0}^n (-1)^k \binom{n}{k} t^{n-k} a^{k-1} (a+kx) = t^n - nxt^{n-1}, \quad n \geq 1 \quad (1.52)$$

$$a^{n-1} \sum_{k=0}^n (-1)^k \binom{n}{k} (a + kx) = \begin{cases} 0, & n \neq 0, 1 \\ -x, & n = 1 \\ 1, & n = 0 \end{cases} \quad (1.53)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (f(x))^k (a + kx) = \quad (1.54)$$

$$(1 - f(x))^{n-1} (a - (a + nx)f(x)), \quad n \geq 1$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (f(x))^k (a - kx) = \quad (1.55)$$

$$(1 - f(x))^{n-1} (a - (a - nx)f(x)), \quad n \geq 1$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{n^k} = \left(1 - \frac{1}{n}\right)^{n-1} \left(a - \left(\frac{a}{n} + x\right)\right), \quad n \geq 1 \quad (1.56)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{n^k} = \frac{a - x}{e} \quad (1.57)$$

$$\sum_{k=0}^n \binom{n}{k} \frac{a + kx}{n^k} = \left(1 + \frac{1}{n}\right)^{n-1} \left(a + \left(\frac{a}{n} + x\right)\right), \quad n \geq 1 \quad (1.58)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k} \frac{a + kx}{n^k} = (a + x)e \quad (1.59)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{2^k} = \frac{a - nx}{2^n} \quad (1.60)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{2^{nk}} = \left(1 - \frac{1}{2^n}\right)^{n-1} \left(a - \frac{a + nx}{2^n}\right), \quad n \geq 1 \quad (1.61)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{2^{nk}} = a \quad (1.62)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{(a + nx)^k} = \left(1 - \frac{1}{a + nx}\right)^{n-1} (a - 1), \quad n \geq 1 \quad (1.63)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a + kx}{(a - nx)^k} = \left(1 - \frac{1}{a - nx}\right)^{n-1} \left(a - \frac{a + nx}{a - nx}\right), \quad n \geq 1 \quad (1.64)$$

$$\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} k (f(x))^k = n f(x) (1 - f(x))^{n-1} \quad (1.65)$$

$$\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} k^2 (f(x))^{k-1} = n (1 - f(x))^{n-1} - n(n-1) f(x) (1 - f(x))^{n-2} \quad (1.66)$$

$$\sum_{k=0}^n \binom{n}{k} k^2 = 2^n \left(\frac{n^2 + n}{4}\right) \quad (1.67)$$

$$\sum_{k=0}^n \binom{n}{k} k^2 2^{k-1} = 3^{n-2} n(2n + 1) \quad (1.68)$$

$$\sum_{k=0}^n \binom{n}{k} a^k k = na(1 + a)^{n-1} \quad (1.69)$$

$$\sum_{k=0}^n \binom{n}{k} k = n2^{n-1} \quad (1.70)$$

$$\sum_{k=0}^n \binom{n}{k} k(k-1)x^{k-2} = n(n-1)(1+x)^{n-2} \quad (1.71)$$

$$\sum_{k=0}^n \binom{n}{k} k(k-1) = n(n-1)2^{n-2} \quad (1.72)$$

1.4.6 Number Theoretic Result Due to Euler

$$\text{Let } f(x) = \sum_{i=0}^n a_i x^i. \quad \text{Then, } f(x) | f(x + f(x)). \quad (1.73)$$

1.5 Four Versions of the Multinomial Theorem

Remark 1.5 *In Section 1.5, we will assume α is a nonnegative integer. We also assume that j_i is a nonnegative integer.*

$$\left(\sum_{i=0}^n a_i \right)^\alpha = \sum_{\substack{\forall j \text{ such that} \\ \sum_{i=0}^n j_i = \alpha}} \frac{\alpha!}{j_0! j_1! j_2! \dots j_n!} a_0^{j_0} a_1^{j_1} \dots a_n^{j_n} \quad (1.74)$$

$$\left(\sum_{i=1}^n a_i \right)^\alpha = \sum_{\substack{\forall j \text{ such that} \\ \sum_{i=1}^n j_i = \alpha}} \frac{\alpha!}{j_1! j_2! \dots j_n!} a_1^{j_1} a_2^{j_2} \dots a_n^{j_n} \quad (1.75)$$

$$\left(\sum_{i=0}^n a_i x^i \right)^\alpha = \sum_{k=0}^{N < \infty} x^k \sum_{\substack{\forall j \text{ such that} \\ \sum_{i=0}^n j_i = \alpha, \sum_{i=1}^n i j_i = k}} \frac{\alpha!}{j_0! j_1! j_2! \dots j_n!} a_0^{j_0} a_1^{j_1} \dots a_n^{j_n} \quad (1.76)$$

$$\left(\sum_{i=0}^n a_i x^i \right)^\alpha = \sum_{k=0}^{N < \infty} x^{k*} \sum_{\substack{\forall j \text{ such that} \\ j_0 + \gamma = \alpha, \sum_{i=1}^n j_i = \gamma, \sum_{i=1}^n i j_i = k}} \binom{\alpha}{\gamma} a_0^{\alpha - \gamma} \frac{\gamma!}{j_1! j_2! \dots j_n!} a_1^{j_1} a_2^{j_2} \dots a_n^{j_n} \quad (1.77)$$

2 The Geometric Series

Remark 2.1 *In this chapter, we will assume, unless otherwise specified, that a is a nonnegative integer and x is an arbitrary nonzero real or complex number.*

2.1 The Basic Geometric Series

2.1.1 Finite Geometric Series

$$\sum_{k=a}^n x^k = x^a \frac{x^{n-a+1} - 1}{x - 1}, \quad x \neq 1 \quad (2.1)$$

$$\sum_{k=a}^n \frac{1}{x^k} = \frac{1}{x^n} \frac{x^{n-a+1} - 1}{x - 1}, \quad x \neq 1 \quad (2.2)$$

2.1.2 Infinite Geometric Series

$$\sum_{k=a}^{\infty} x^k = \frac{x^a}{1 - x}, \quad |x| < 1 \quad (2.3)$$

$$\sum_{k=a}^{\infty} x^{-k} = \frac{x^{1-a}}{x - 1}, \quad |x| > 1 \quad (2.4)$$

2.2 Derivatives of Geometric Series

$$\sum_{k=0}^n kx^{k-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}, \quad x \neq 1 \quad (2.5)$$

$$\lim_{x \rightarrow 1} \sum_{k=0}^n kx^k = \frac{n^2 + n}{2} = \sum_{k=0}^n k \quad (2.6)$$

$$\sum_{k=0}^n k^2 x^k = \frac{n^2 x^{n+3} - (2n^2 + 2n - 1)x^{n+2} + (n+1)^2 x^{n+1} - x^2 - x}{(x-1)^3}, \quad x \neq 1 \quad (2.7)$$

$$\sum_{k=0}^n k(k-1)2^k = (n^2 - 3n + 4)2^{n+1} - 2^3 \quad (2.8)$$

$$\sum_{k=0}^n k^2 2^k = (n^2 - 2n + 3)2^{n+1} - 6 \quad (2.9)$$

$$\sum_{k=0}^n k^2 3^k = \frac{(n^2 - n + 1)3^{n+1} - 3}{2} \quad (2.10)$$

2.3 Integrals of Geometric Series

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, \quad |x| < 1 \quad (2.11)$$

$$\ln 2 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \quad (2.12)$$

2.4 Applications of Geometric Series

Remark 2.2 *In the following two identities, let u and v be arbitrary nonzero real or complex numbers such that $uv \neq 1$.*

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (uv)^k + u \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} (uv)^k = \frac{(uv)^{\lfloor \frac{n+1}{2} \rfloor} - 1}{uv - 1} + u \frac{(uv)^{\lfloor \frac{n}{2} \rfloor} - 1}{uv - 1}, \quad n \geq 1 \quad (2.13)$$

Remark 2.3 *The following identity can be done as a formal calculation over the ring of power series. Otherwise, the reader may assume that appropriate condition hold so that the left sum is absolutely convergent.*

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{i=0}^k 2^i f(i) \quad (2.14)$$

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1 - x^{2^{k+1}}} = \frac{1}{1 - x}, \quad |x| > 1 \quad (2.15)$$

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1 - x^{2^{k+1}}} = \frac{x}{1 - x}, \quad |x| < 1 \quad (2.16)$$

3 Bernoulli-Type Series and the Riemann Zeta Function

Remark 3.1 *In this chapter, we will assume p and a are, unless otherwise specified, nonnegative integers.*

3.1 Evaluation of $\sum_{k=1}^n k^p$

3.1.1 Reduction Formula

$$\sum_{k=a}^n k^p = n \sum_{k=a}^n k^{p-1} - \sum_{r=a}^{n-1} \sum_{k=a}^r k^{p-1}, \quad p \geq 1, \quad n \geq 1 \quad (3.1)$$

3.1.2 Iteration Formulas

Remark 3.2 *In this subsection, we let r be a positive integer, and define $\sum_{(r)}^n f(k)$ to be the following r -fold sum:*

$$\sum_{(r)}^n f(k) = \sum_{k_r=1}^n \sum_{k_{r-1}=1}^{k_r} \dots \sum_{k_2=1}^{k_3} \sum_{k=1}^{k_2} f(k) >$$

$$\sum_{(r)}^n 1 = \binom{n+r-1}{r} \quad (3.2)$$

$$\sum_{(r)}^n k = \binom{n+r}{r+1} \quad (3.3)$$

$$\sum_{(1)}^n k = \frac{n(n+1)}{2} \quad (3.4)$$

$$\sum_{(r)}^n k^2 = \frac{(2n+r)(n+r)!}{(r+2)!(n-1)!} \quad (3.5)$$

$$\sum_{(1)}^n k^2 = \frac{n(n+1)(2n+1)}{3!} \quad (3.6)$$

$$\sum_{(r)}^n k^3 = \frac{6n^2 + r(6n + r - 1)}{(r + 3)!} \frac{(n + r)!}{(n - 1)!} \quad (3.7)$$

$$\sum_{(1)}^n k^3 = \frac{n^2(n + 1)^2}{4} \quad (3.8)$$

$$\sum_{(r)}^n k^4 = \frac{(12n^2 + 12rn - r(5 - r))(2n + r)}{(r + 4)!} \frac{(n + r)!}{(n - 1)!} \quad (3.9)$$

$$\sum_{(1)}^n k^4 = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30} \quad (3.10)$$

$$\sum_{(1)}^n k^5 = \frac{n^2(n + 1)^2(2n^2 + 2n - 1)}{12} \quad (3.11)$$

$$\sum_{(1)}^n k^6 = \frac{n(n + 1)(2n + 1)(3n^4 + 6n^3 - 3n + 1)}{42} \quad (3.12)$$

3.1.3 Euler's Expansion with Bernoulli Numbers

Remark 3.3 In this subsection, we let \mathcal{B}_k denote the k^{th} Bernoulli number. The exponential generating function of $(\mathcal{B}_k)_{n=0}^{\infty}$ is $\frac{x}{e^x - 1}$. For values of the Bernoulli number sequence, the reader is referred to the Online Encyclopedia of Integer Sequences (OEIS).

$$\sum_{k=0}^{n-1} k^p = \sum_{k=0}^p \frac{p! n^{p-k+1}}{(p - k + 1)! k!} \mathcal{B}_k, \quad n \geq 1 \quad (3.13)$$

3.1.4 G. P. Miller's Determinant Expansion

$$\sum_{k=0}^n k^p = \frac{\det M}{(p + 1)!}, \quad (3.14)$$

where M is the $(p + 1) \times (p + 1)$ matrix whose entry $a_{i,j}$ is determined by

$$a_{i,j} = \begin{cases} (n + 1)^{p+2-i} - (n + 1), & j = 1 \\ \binom{p+2-i}{j+1-i}, & j \geq i \text{ and } j \neq 1 \\ 0, & j < i \text{ and } j \neq 1 \end{cases} \quad (3.15)$$

3.2 Evaluation of $\sum_{k=0}^n k^p x^k$

Remark 3.4 *The reader should compare the formulas in this subsection with those in Section 2.2.*

3.2.1 Differential Reduction Formula

$$\sum_{k=0}^n k^{p+1} x^k = x \frac{d}{dx} \sum_{k=0}^n k^p x^k \quad (3.16)$$

3.2.2 Applications of Differential Reduction Formula

$$\sum_{k=0}^n k^3 x^k = \frac{N}{(x-1)^4}, \quad \text{where for } x \neq 1, \quad (3.17)$$

$$N = n^3 x^{n+4} - (3n^3 + 3n^2 - 3n + 1)x^{n+3} + (3n^3 + 6n^2 - 4)x^{n+2} - (n+1)^3 x^{n+1} + x^3 + 4x^2 + x$$

$$\sum_{k=1}^{n-1} k \left(\frac{n}{n-1} \right)^{k-1} = (n-1)^2, \quad n \geq 1 \quad (3.18)$$

$$\prod_{k=2}^n \left(1 + \frac{\binom{n+1}{n}^{k-1}}{k - \binom{k+1}{k}^{k-2}} \right) = n^2, \quad n \geq 2 \quad (3.19)$$

$$\sum_{k=1}^n \frac{k}{n^{k-1}} (n+1)^{k-1} = \sum_{k=1}^n \frac{k}{n^{k-1}} \binom{n+1}{k+1} = n^2 \quad n \geq 1 \quad (3.20)$$

3.3 Evaluation of $\sum_{k=0}^n \binom{n}{k} k^p$

$$\sum_{r=0}^n \binom{n}{r} = 2^n \quad (3.21)$$

$$\sum_{r=0}^n \binom{n}{r} r = n 2^{n-1} \quad (3.22)$$

$$\sum_{r=0}^n \binom{n}{r} r^2 = 2^{n-2} n(n+1) \quad (3.23)$$

$$\sum_{r=0}^n \binom{n}{r} r^3 = 2^{n-3} n^2 (n+3) \quad (3.24)$$

$$\sum_{r=0}^n \binom{n}{r} r^4 = 2^{n-4} n (n+1) (n^2 + 5n - 2) \quad (3.25)$$

3.3.1 Reduction Formula

$$n \sum_{r=0}^{n-1} \binom{n-1}{r} r^p = \sum_{k=0}^p \sum_{r=0}^n (-1)^k \binom{p}{k} \binom{n}{r} r^{p-k+1}, \quad n \geq 1 \quad (3.26)$$

3.4 Riemann Zeta Function: $\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}$

3.4.1 Convolution Identity

Remark 3.5 *The following identity is found in “A New Method of Evaluating $\zeta(2n)$ ”, by G.T. Williams, Amer. Math. Monthly, January 1953, Vol. 60, No. 1, pp. 19-25.*

$$\sum_{k=2}^{n-1} \zeta(k) \zeta(n-k+1) = (n+2) \zeta(n+1) - 2 \sum_{k=1}^{\infty} \frac{1}{k^n} \sum_{j=1}^k \frac{1}{j}, \quad n \geq 3 \quad (3.27)$$

Extension of Convolution Identity

$$4\zeta(3) - 2 \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^k \frac{1}{j} = 0 \quad (3.28)$$

3.4.2 Connections with Bernoulli Numbers

Remark 3.6 *Recall that \mathcal{B}_n is the n^{th} Bernoulli number. See Remark 3.3.*

$$\mathcal{B}_{2n} = (-1)^{n-1} \frac{(2n)!}{2^{2n-1} \pi^{2n}} \zeta(2n) \quad (3.29)$$

$$\sum_{k=1}^{n-1} \zeta(2k) \zeta(2n-2k) = \left(n + \frac{1}{2}\right) \zeta(2n), \quad n \geq 2 \quad (3.30)$$

4 Finite Harmonic Series

4.1 Special Case of n^{th} Difference Inversion Formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+1} = \frac{1}{n+1} \quad (4.1)$$

$$\sum_{j=1}^n \frac{1}{j} = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{1}{k}, \quad n \geq 1 \quad (4.2)$$

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{j=1}^k \frac{1}{j} = \frac{1}{n}, \quad n \geq 1 \quad (4.3)$$

4.2 Even and Odd Finite Harmonic Series

$$\sum_{k=1}^{2n} \frac{1}{k} = \frac{2n+1}{2} \sum_{k=1}^{2n} \frac{1}{k(2n-k+1)} = (2n+1) \sum_{k=1}^n \frac{1}{k(2n-k+1)}, \quad n \geq 1 \quad (4.4)$$

$$\sum_{k=1}^{2n+1} \frac{1}{k} = (2n+2) \sum_{k=1}^n \frac{1}{k(2n-k+2)} + \frac{1}{n+1} \quad (4.5)$$

4.3 Harmonic Series as Limit of Binomial Coefficient

Remark 4.1 *In the following identity, we let r be a positive integer.*

$$\sum_{k=1}^n \frac{1}{k} = \lim_{r \rightarrow 0} \frac{\binom{n+r}{r} - 1}{r \binom{n+r}{r}} \quad (4.6)$$

Fundamentals of Series: Table III: Basic Algebraic Techniques

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 Telescoping Series

1.1 Two Basic Telescoping Identities

Remark 1.1 *In this chapter, we assume, unless otherwise specified, that a is a nonnegative integer. We also assume that x and r are arbitrary real or complex numbers, unless otherwise specified.*

First Telescoping Identity

Let $u_k = f(k) - f(k + 1)$. Then,

$$\sum_{k=a}^n u_k = f(a) - f(n + 1), \quad n \geq a \quad (1.1)$$

Second Telescoping Identity

Let $v_k = f(k) + f(k + 1)$. Then,

$$\sum_{k=a}^n (-1)^k v_k = (-1)^a f(a) - (-1)^{n+1} f(n + 1), \quad n \geq a \quad (1.2)$$

1.1.1 Applications of Basic Telescoping Identities

$$\sum_{k=a}^{n-1} \frac{r^k}{(1 + r^k x)(1 + r^{k+1} x)} = \frac{1}{x(r - 1)} \left(\frac{1}{1 + r^a x} - \frac{1}{1 + r^n x} \right), \quad (1.3)$$

$n \geq a + 1, r \neq 1, x \neq 0$

$$\sum_{k=1}^{n-1} \frac{r^{k-1}}{(1-r^k)(1-r^{k+1})} = \frac{1}{r(1-r)} \left(\frac{1}{1-r} - \frac{1}{1-r^n} \right), \quad r \neq 1 \quad (1.4)$$

$$\sum_{k=1}^{\infty} \frac{r^{k-1}}{(1-r^k)(1-r^{k+1})} = \begin{cases} \frac{1}{(1-r)^2}, & |r| < 1 \\ \frac{1}{r(1-r)^2}, & |r| > 1 \end{cases} \quad (1.5)$$

$$\sum_{k=a}^{n-1} \frac{1}{(r+kx)(r+(k+1)x)} = \frac{1}{x} \left(\frac{1}{r+ax} - \frac{1}{r+nx} \right), \quad n \geq a+1, x \neq 0 \quad (1.6)$$

$$\sum_{k=1}^n \frac{1}{(r+kx)(r+(k+1)x)} = \frac{n}{(r+x)(r+(n+1)x)} \quad (1.7)$$

Hockey Stick Identity

$$\sum_{k=1}^n \binom{k-1}{m-1} = \binom{n}{m}, \text{ where } m \text{ is a positive integer and } n \geq 1 \quad (1.8)$$

$$\sum_{k=a}^n (-1)^k \binom{x}{k} = (-1)^a \binom{x-1}{a-1} + (-1)^n \binom{x-1}{n}, \quad n \geq a \quad (1.9)$$

$$\sum_{k=0}^n \binom{x-k}{n-k} = \binom{x+1}{n} \quad (1.10)$$

Remark 1.2 *In the following identity, we interpret $x!$ by the Gamma function, namely $\Gamma(x+1) = x!$, whenever x is not a negative integer.*

$$\sum_{j=a}^n \binom{j}{x} = \binom{n+1}{x+1} - \binom{a}{x+1}, \quad n \geq a \quad (1.11)$$

$$\sum_{k=0}^n 2^{n-k} \binom{x+k}{k} \frac{x-k}{x+k} = \binom{x+n}{n}, \quad x \neq 0 \quad (1.12)$$

Remark 1.3 *The following identity, proposed by M. S. Klamkin, is Problem 4561, p. 632 of the November 1953 (Vol. 60, No. 9) issue of Amer. Math. Monthly.*

$$\sum_{k=1}^n \frac{1}{k^2} \left(1 - \frac{1}{\binom{k+r}{k}} \right) = \sum_{k=1}^r \frac{1}{k^2} \left(1 - \frac{1}{\binom{k+n}{k}} \right), \quad r \text{ and } n \text{ positive integers} \quad (1.13)$$

$$\sum_{k=1}^n \frac{1}{2k} \frac{2^{2k}}{\binom{2k}{k}} = \sum_{k=1}^n \frac{(-1)^k}{2k \binom{-1}{k}} = \frac{2^{2n}}{\binom{2n}{n}} - 1, \quad n \geq 1 \quad (1.14)$$

$$\sum_{k=1}^n \binom{2k-2}{k} \frac{1}{k+1} = \frac{1}{3} \binom{2n}{n} \frac{1}{n+1} - \frac{1}{3}, \quad n \geq 1 \quad (1.15)$$

$$\sum_{k=1}^n \frac{1}{(\sqrt{x+k} + \sqrt{x+k+1})\sqrt{(x+k)(x+k+1)}} = \frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x+n+1}} \quad (1.16)$$

$$\sum_{k=1}^n \frac{1}{(\sqrt{k} + \sqrt{k+1})\sqrt{k(k+1)}} = 1 - \frac{1}{\sqrt{n+1}}, \quad n \geq 1 \quad (1.17)$$

$$\sum_{k=1}^{\infty} \frac{1}{(\sqrt{k} + \sqrt{k+1})\sqrt{k(k+1)}} = 1 \quad (1.18)$$

$$\sum_{k=0}^{\infty} \frac{1}{(\sqrt{x+k} + \sqrt{x+k+1})\sqrt{(x+k)(x+k+1)}} = \frac{1}{\sqrt{x}} \quad (1.19)$$

2 Summing Series by Recursive Formulas

2.1 Product Functions and Recursive Formulas

2.1.1 Product Function $f(n) = \prod_{k=1}^n \frac{1}{2k+1}$

Remark 2.1 Note that $f(n) = \frac{1}{(2n+1)!!}$. For more information about double factorials, see the Wikipedia entry.

$$\sum_{k=1}^n \frac{k}{\prod_{j=1}^k (2j+1)} = \sum_{k=1}^n \frac{k k! 2^k}{(2k+1)!} = \frac{1}{2} \left(1 - \frac{1}{\prod_{k=1}^n (2k+1)} \right), \quad n \geq 1 \quad (2.1)$$

$$\sum_{k=1}^{\infty} \frac{k}{\prod_{j=1}^k (2j+1)} = \sum_{k=1}^{\infty} \frac{k k! 2^k}{(2k+1)!} = \frac{1}{2} \quad (2.2)$$

Remark 2.2 In the following two identities, we evaluate any non-integer factorials by the Gamma function, i.e. $x! = \Gamma(x+1)$, whenever x is not a negative integer.

$$\sum_{k=1}^{\infty} \frac{k}{2^k (k + \frac{1}{2})!} = \frac{1}{\sqrt{\pi}} \quad (2.3)$$

$$\sum_{k=0}^{\infty} \frac{k}{2^k (k + \frac{1}{2})!} = \sqrt{\pi} \quad (2.4)$$

$$\sum_{k=0}^{\infty} \frac{(k!)^2 2^k}{(2k+1)!} = \frac{\pi}{2} \quad (2.5)$$

2.1.2 Product Function $h(n) = \prod_{k=1}^n 2k$

Remark 2.3 Note that $h(n) = (2n)!!$.

$$\sum_{k=1}^n (2k-1) 2^{k-1} (k-1)! = 2^n n! - 1, \quad n \geq 1 \quad (2.6)$$

2.1.3 Product Function $g(n) = \prod_{k=1}^n \frac{1}{2k}$

Remark 2.4 Note that $g(n) = \frac{1}{(2n)!!}$.

$$\sum_{k=1}^n \frac{2k-1}{2^k k!} = 1 - \frac{1}{2^n n!}, \quad n \geq 1 \quad (2.7)$$

$$\sum_{k=0}^n \frac{1-2k}{2^k k!} = \frac{1}{2^n n!} \quad (2.8)$$

$$\sum_{k=1}^{\infty} \frac{2k-1}{2^k k!} = 1 \quad (2.9)$$

2.1.4 Product Function $F(n) = \prod_{k=1}^n (2k+1)^p$

Remark 2.5 For $F(n)$, we normally assume p is a nonnegative integer. However, we should note that the first identity in this subsection holds if p is any positive real number.

$$\sum_{k=1}^n ((2k+1)^p - 1) \frac{(2k-1)!^p}{(k-1)!^p 2^{pk-p}} = \frac{(2n+1)!^p}{n!^p 2^{pn}} - 1, \quad n \geq 1 \quad (2.10)$$

$$\sum_{k=1}^n \frac{k(2k-1)!}{(k-1)! 2^{k-1}} = \frac{(2n+1)!}{n! 2^{n+1}} - \frac{1}{2}, \quad n \geq 1 \quad (2.11)$$

$$\sum_{k=1}^n \binom{2k-1}{k} \frac{k^2(k+1)}{2^{2k}} (2k-1)! = \frac{(2n+1)!^2}{n!^2 2^{2n}} - 1, \quad n \geq 1 \quad (2.12)$$

$$\sum_{k=1}^n \sum_{j=1}^p \binom{p}{j} (2k)^j \left(\frac{(2k-1)!}{(k-1)! 2^{k-1}} \right)^p = \left(\frac{(2n+1)!}{n! 2^n} \right)^p - 1 \quad (2.13)$$

2.1.5 Product Function $G(n) = \prod_{k=1}^n (2k + 1)^{-p}$

Remark 2.6 For $G(n)$, we assume p is a nonnegative integer.

$$\sum_{k=1}^n \sum_{j=1}^p \binom{p}{j} (2k)^j \left(\frac{k!2^k}{(2k+1)!} \right)^p = 1 - \left(\frac{n!2^n}{(2n+1)!} \right)^p \quad (2.14)$$

$$\sum_{k=1}^{\infty} \sum_{j=1}^p \binom{p}{j} (2k)^j \left(\frac{k!2^k}{(2k+1)!} \right)^p = 1 \quad (2.15)$$

2.2 Evaluation of $\sum_{k=0}^n \binom{n}{k}^q$

Remark 2.7 The formulas in this section can be found in Tor. B. Staver's "Om summasjon av potenser av binomialkoeffisienten", Norsk Matematisk Tidsskrift, 29.Årgang, 1947, pp. 97-103.

2.2.1 Let $q = -1$

$$\sum_{k=0}^n \frac{k}{\binom{n}{k}} = \frac{n}{2} \sum_{k=0}^n \frac{1}{\binom{n}{k}} \quad (2.16)$$

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}} = \frac{n+1}{2n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} + 1 \quad (2.17)$$

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{j=1}^{n+1} \frac{2^j}{j} \quad (2.18)$$

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}} = \frac{(n+1)}{S_{n+1}(1)} \sum_{k=1}^{n+1} \frac{S_k(1)}{k}, \text{ where } S_n(1) = \sum_{k=0}^n \binom{n}{k} = 2^n \quad (2.19)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{\binom{n}{k}} = 2 \quad (2.20)$$

$$\lim_{n \rightarrow \infty} \frac{n}{2^n} \sum_{k=1}^n \frac{2^k}{k} = 2 \quad (2.21)$$

2.2.2 Let $q = -2$

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}^2} = \frac{(n+2)^3}{2(2n+5)(n+1)^2} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}^2} + \frac{3(n+2)}{2n+5} \quad (2.22)$$

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}^2} = \frac{3(n+1)^2}{2n+3} \frac{1}{\binom{2n+2}{n+1}} \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k} \quad (2.23)$$

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}^2} = \frac{3(n+1)^2}{2n+3} \frac{1}{S_{n+1}(2)} \sum_{k=1}^{n+1} \frac{S_k(2)}{k}, \quad \text{where } S_n(2) = \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad (2.24)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{\binom{n}{k}^2} = 2 \quad (2.25)$$

$$\lim_{n \rightarrow \infty} \frac{3(n+1)^2}{2n+3} \frac{1}{\binom{2n+2}{n+1}} \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k} = 2 \quad (2.26)$$

2.2.3 Evaluation of $\sum_{k=0}^n \frac{x^k}{\binom{n}{k}}$

$$\sum_{k=0}^n \frac{x^k}{\binom{n}{k}} = (n+1) \left(\frac{x}{1+x} \right)^{n+1} \sum_{k=1}^{n+1} \frac{1+x^k}{k(1+x)} \left(\frac{1+x}{x} \right)^k, \quad x \neq -1 \quad (2.27)$$

$$\left(1 + \frac{1}{x} \right) \sum_{k=0}^{n+1} \frac{x^k}{\binom{n+1}{k}} = \frac{n+2}{n+1} \sum_{k=0}^n \frac{x^k}{\binom{n}{k}} + x^{n+1} + \frac{1}{x}, \quad x \neq 0 \quad (2.28)$$

3 Rational Functions and Partial Fraction Decompositions

Remark 3.1 *In this chapter, we assume, unless otherwise specified, that r is a positive integer.*

3.1 Evaluation of $\sum_{k=1}^n \frac{1}{k(k+1)\dots(k+r)}$

$$\sum_{k=1}^n \frac{1}{\prod_{i=0}^r (k+i)} = \frac{1}{r} \left(\frac{1}{r!} - \frac{1}{\prod_{j=1}^r (n+j)} \right), \quad n, r \geq 1 \quad (3.1)$$

3.1.1 Applications of Equation (3.1)

$$\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}, \quad n \geq 1 \quad (3.2)$$

$$\sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right), \quad n \geq 1 \quad (3.3)$$

$$\sum_{k=1}^n \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3} \left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)} \right), \quad n \geq 1 \quad (3.4)$$

$$\sum_{k=1}^n \frac{1}{\prod_{i=0}^r (k+i)} = \sum_{k=1}^n \sum_{t=0}^r \frac{(-1)^t}{t!(r-t)!(k+t)}, \quad n, r \geq 1 \quad (3.5)$$

$$\sum_{k=1}^n \sum_{t=0}^r (-1)^t \binom{r}{t} \frac{1}{k+t} = \frac{1}{r} - n \sum_{t=0}^r (-1)^t \binom{r}{t} \frac{1}{n+t}, \quad n, r \geq 1 \quad (3.6)$$

3.1.2 Connections to the Gamma Function

Remark 3.2 *Let p and q be any real or complex numbers which are not negative integers. Then,*

$$\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)!}{(p+q-1)!}, \quad (3.7)$$

is an expression for the Beta Function.

$$\beta(n+1, m+1) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{m+k+1} \quad (3.8)$$

$$\sum_{k=1}^n \beta(r+1, k) = \frac{1}{r} - n\beta(r+1, n), \quad n \geq 1 \quad (3.9)$$

3.2 Evaluation of $\sum_{k=1}^n \frac{1}{(k+a)(k+a+1)\dots(k+a+r)}$

Remark 3.3 In Section 3.2, we assume a is any nonzero real or complex number which is not a negative integer. Note that $a! = \Gamma(a+1)$.

$$\begin{aligned} \sum_{k=1}^n \prod_{i=0}^r \frac{1}{k+a+i} &= \sum_{k=1}^n \frac{(k+a-1)!}{(k+a+r)!} \\ &= \frac{1}{r} \left(\frac{a!}{(a+r)!} - \frac{(n+a)!}{(n+a+r)!} \right), \quad n, r \geq 1 \end{aligned} \quad (3.10)$$

3.2.1 Applications of Equation (3.10)

$$\sum_{k=1}^n \frac{1}{(k+a)(k+a+1)} = \frac{n}{(a+1)(n+a+1)}, \quad n \geq 1 \quad (3.11)$$

$$\sum_{k=1}^n \frac{1}{(k+a)(k+a+1)(k+a+2)} = \frac{1}{2} \left(\frac{1}{(a+1)(a+2)} - \frac{1}{(n+a+1)(n+a+2)} \right) \quad (3.12)$$

$$\sum_{k=0}^n \binom{k+a}{r} = \binom{n+a+1}{r+1} - \binom{a}{r+1} \quad (3.13)$$

$$\sum_{k=0}^n \binom{k-a}{r} = \binom{n-a+1}{r+1} + (-1)^r \binom{a+r}{r+1} \quad (3.14)$$

$$\sum_{k=0}^n \frac{(k+n)!}{k!} = \frac{(2n+1)!}{(n+1)!} \quad (3.15)$$

$$\sum_{k=0}^n \frac{(k+a)!}{k!} = \frac{(n+a+1)!}{(a+1)n!} \quad (3.16)$$

$$\sum_{k=0}^n (k+1) = \frac{(n+2)(n+1)}{2} \quad (3.17)$$

$$\sum_{k=0}^n (k+1)(k+2) = \frac{(n+3)(n+2)(n+1)}{3} \quad (3.18)$$

$$\sum_{k=0}^n \binom{k+n}{k} = \binom{2n+1}{n+1} \quad (3.19)$$

$$\sum_{k=0}^n \binom{k+a}{k} = \binom{n+a+1}{n} \quad (3.20)$$

$$\sum_{k=0}^n \binom{3n-k}{2n} = \binom{3n+1}{n} \quad (3.21)$$

3.2.2 Connections to Pascal's Formulas

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad n \geq 1 \quad (3.22)$$

$$\sum_{k=1}^n \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}, \quad n \geq 1 \quad (3.23)$$

$$\sum_{k=1}^n \frac{k(k+1)(k+2)}{6} = \frac{n(n+1)(n+2)(n+3)}{24}, \quad n \geq 1 \quad (3.24)$$

Pascal's Formula

$$\sum_{k=1}^n \binom{k+r-1}{k-1} = \binom{n+r}{r+1}, \quad \text{where } r \text{ is a nonnegative integer} \quad (3.25)$$

3.2.3 Reciprocal Applications of Equation (3.10)

$$\sum_{k=1}^n \frac{1}{\binom{k+a+r}{r+1}} = \left(1 + \frac{1}{r}\right) \left(\frac{1}{\binom{a+r}{r}} - \frac{1}{\binom{n+a+r}{r}}\right), \quad n \geq 1 \quad (3.26)$$

$$\sum_{k=0}^n \frac{1}{\binom{k+a+r}{r+1}} = \left(1 + \frac{1}{r}\right) \left(\frac{1}{\binom{a+r-1}{r}} - \frac{1}{\binom{n+a+r}{r}}\right) \quad (3.27)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+a+r}{r+1}} = \left(1 + \frac{1}{r}\right) \frac{1}{\binom{a+r}{r}} \quad (3.28)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r+1}{r+1}} = \frac{1}{r} \quad (3.29)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2n+k}{n+1}} = \left(1 + \frac{1}{n}\right) \frac{1}{\binom{2n}{n}} = \frac{1}{\binom{2n}{n-1}}, \quad n \geq 1 \quad (3.30)$$

$$\sum_{k=1}^n \frac{1}{\binom{2n+k}{n}} = \frac{n}{n-1} \left(\frac{1}{\binom{2n}{n-1}} - \frac{1}{\binom{3n}{n-1}}\right), \quad n \geq 1 \quad (3.31)$$

$$\sum_{k=1}^n \frac{1}{\binom{k+r}{k}} = \left(1 + \frac{1}{r-1}\right) \left(\frac{1}{r} - \frac{1}{\binom{n+r}{r-1}}\right), \quad n \geq 1 \quad (3.32)$$

$$\sum_{k=0}^n \frac{1}{\binom{k+r}{k}} = \left(1 + \frac{1}{r-1}\right) \left(1 - \frac{1}{\binom{n+r}{r-1}}\right) \quad (3.33)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r}{k}} = \sum_{k=1}^{\infty} \frac{1}{\prod_{j=1}^r \left(1 + \frac{k}{j}\right)} = \frac{1}{r-1} \quad (3.34)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r}{k} k} = \frac{1}{r} \quad (3.35)$$

$$\sum_{k=0}^{\infty} \frac{1}{\binom{a+k}{k}} = \frac{a}{a-1}, \quad a \neq 1 \quad (3.36)$$

Finite Harmonic Series Formula

$$\sum_{j=1}^n \frac{1}{j} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=1}^n \frac{1}{\binom{j+k}{k}}, \quad n \geq 1 \quad (3.37)$$

3.3 Evaluation of $\sum_{k=a}^n \frac{1}{(k-x)(k-x+1)}$

Remark 3.4 *In Section 3.3, we assume a is any nonnegative integer and x is a nonzero real or complex number for which the series are defined.*

$$\sum_{k=a}^n \frac{1}{(k-x)(k-x+1)} = \frac{1}{a-x} - \frac{1}{n-x+1} \quad (3.38)$$

3.3.1 Derivatives of Equation (3.38)

$$\sum_{k=a}^n \frac{2k-2x+1}{(k-x)^2(k-x+1)^2} = \frac{1}{(a-x)^2} - \frac{1}{(n-x+1)^2} \quad (3.39)$$

$$\sum_{k=a}^n \frac{2k+1}{k^2(k+1)^2} = \frac{1}{a^2} - \frac{1}{(n+1)^2}, \quad a \neq 0 \quad (3.40)$$

$$\sum_{k=a}^n \frac{2k+3}{(k+1)^2(k+2)^2} = \frac{1}{(a+1)^2} - \frac{1}{(n+2)^2} \quad (3.41)$$

3.4 Evaluation of $\sum_{k=1}^n \frac{1}{k(k+r)}$

$$\sum_{k=1}^n \frac{1}{k(k+r)} = \frac{1}{r} \sum_{i=1}^r \frac{n}{i(i+n)}, \quad n \geq 1 \quad (3.42)$$

3.4.1 Applications of Equation (3.42)

$$\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}, \quad n \geq 1 \quad (3.43)$$

$$\sum_{k=1}^n \frac{1}{k(k+2)} = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right), \quad n \geq 1 \quad (3.44)$$

$$\sum_{k=1}^n \frac{1}{k(k+3)} = \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right), \quad n \geq 1 \quad (3.45)$$

Harmonic Series Formula

$$\sum_{r=1}^n \frac{1}{r} = n \sum_{k=1}^{\infty} \frac{1}{k(k+n)}, \quad n \geq 1 \quad (3.46)$$

Generalization of Equation (3.42)

$$\sum_{k=a}^n \frac{1}{k(k+r)} = \frac{1}{r} \sum_{i=1}^r \frac{n-a+1}{(a+i-1)(n+i)}, \quad \text{where } a \text{ is a positive integer} \quad (3.47)$$

3.5 Evaluation of $\sum_{k=1}^n \frac{r}{(b+k)(b+k+r)}$

Remark 3.5 *In Section 3.5, we assume b is a positive integer.*

$$\sum_{k=1}^n \frac{r}{(b+k)(b+k+r)} = \sum_{k=1}^r \frac{n}{(b+k)(b+k+n)}, \quad n \geq 1 \quad (3.48)$$

3.5.1 Applications of Equation (3.48)

$$\sum_{k=0}^{n-1} \frac{1}{(b+k)(b+k+1)} = \frac{n}{b(n+b)}, \quad n \geq 1 \quad (3.49)$$

Remark 3.6 *In the following identity, we assume p and q are positive integers.*

$$\sum_{k=0}^{n-1} \frac{1}{(p+qk)(p+q(k+1))} = \frac{n}{p(p+qn)}, \quad n \geq 1 \quad (3.50)$$

3.6 Evaluation of $\sum \frac{1}{n^2 - a^2}$

Remark 3.7 In Section 3.6, we assume a is a positive integer.

$$\sum_{n=a+1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a} \sum_{k=1}^{2a} \frac{1}{k} \quad (3.51)$$

$$\sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{n^2 - k^2} = \frac{-3}{4n^2}, \quad n \geq 1 \quad (3.52)$$

4 Pascal's Identity in Evaluation of Series

Remark 4.1 In this chapter, we assume, unless otherwise specified, that x is an arbitrary real or complex number. We will also assume k is a nonnegative integer, while r is a positive integer.

4.1 Iterations of Pascal's Identity

4.1.1 Calculations Involving $\sum_{t=0}^{r-1} (-1)^t \binom{x+1}{k-t}$

$$\sum_{t=0}^{r-1} (-1)^t \binom{x+1}{k-t} = \binom{x}{k} - (-1)^r \binom{x}{k-r} \quad (4.1)$$

$$\sum_{t=0}^{\infty} (-1)^t \binom{x+1}{k-t} = \binom{x}{k} \quad (4.2)$$

$$\sum_{t=0}^{n-1} (-1)^t \binom{n+1}{n-t} = 1 - (-1)^n \quad (4.3)$$

4.1.2 Calculations Involving $\sum_{t=1}^r (-1)^{t-1} \binom{x+1}{k+t}$

$$\sum_{t=1}^r (-1)^{t-1} \binom{x+1}{k+t} = \binom{x}{k} - (-1)^r \binom{x}{k+r} \quad (4.4)$$

$$\sum_{t=0}^r (-1)^t \binom{x+1}{k+t} = \binom{x}{k-1} + (-1)^r \binom{x}{k+r} \quad (4.5)$$

$$\sum_{t=0}^r (-1)^t \binom{x+1}{t} = (-1)^r \binom{x}{r} \quad (4.6)$$

4.1.3 Calculations Involving $\sum_{t=0}^{r-1} \binom{x+t}{k-1}$

$$\sum_{t=0}^{r-1} \binom{x+t}{k-1} = \binom{x+r}{k} - \binom{x}{k}, \quad k \geq 1 \quad (4.7)$$

$$\sum_{t=0}^{n-1} \binom{n+t}{k-1} = \binom{2n}{k} - \binom{n}{k} \quad (4.8)$$

Remark 4.2 *The following identity is found in “The class of the free metabelian group with exponent p^2 ”, by S. Bachmuth and H. Y. Mochizuki, Communications on Pure and Applied Math., Vol. 21 (1968), pp. 385-399.*

$$\sum_{k=0}^n \binom{k}{i} \binom{k}{j} = \sum_{k=0}^i \binom{i}{k} \binom{j}{k} \binom{n+k+1}{i+j+1}, \quad 0 \leq i \leq j \quad (4.9)$$

4.1.4 Calculations Involving $\sum_{k=1}^n \binom{2x}{2k}$

Remark 4.3 *In this subsection, we assume p is a nonnegative integer. We also assume $[x]$ is the greatest integer in x .*

$$\sum_{k=1}^p \binom{2x}{2k} = \sum_{k=1}^{2p} \binom{2x-1}{k} \quad (4.10)$$

$$\sum_{k=0}^n \binom{2n}{2k} = \sum_{k=0}^{2n} \binom{2n-1}{k} = \begin{cases} 2^{2n-1}, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad (4.11)$$

$$\sum_{k=0}^n \binom{2n}{2k} = \sum_{k=0}^n \binom{2n}{2k+1} = 2^{2n-1}, \quad n \neq 0 \quad (4.12)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} = 2^{n-1}, \quad n \geq 1 \quad (4.13)$$

$$\sum_{k=0}^n \binom{2n+1}{2k+1} = 2^{2n} \quad (4.14)$$

$$\sum_{k=0}^n \binom{2n+1}{2k+1} k = (2n-1)2^{2n-2}, \quad n \geq 1 \quad (4.15)$$

$$\sum_{k=0}^n \binom{2n+1}{2k} = \sum_{k=0}^n \binom{2n+1}{2k+1} = 2^{2n} \quad (4.16)$$

4.2 Generalizations of Equation (4.1)

$$\sum_{j=0}^r (-1)^j \binom{x}{r-j} = \binom{x-1}{r}, \quad r \geq 0 \quad (4.17)$$

$$\sum_{j=0}^r (-1)^j \binom{x}{j} = (-1)^r \binom{x-1}{r}, \quad r \geq 0 \quad (4.18)$$

$$\sum_{k=0}^n (-1)^k \binom{x}{k} = \prod_{k=1}^n \left(1 - \frac{x}{k}\right) = (-1)^n \binom{x-1}{n} = \binom{n-x}{n} \quad (4.19)$$

4.2.1 Applications of Equation (4.19)

$$\sum_{k=1}^n (-1)^k \binom{x}{k} k = (-1)^n x \binom{x-2}{n-1}, \quad n \geq 1 \quad (4.20)$$

With the $\frac{1}{2}$ Transformation

$$\sum_{k=0}^n (-1)^k \binom{\frac{1}{2}}{k} = (-1)^n \binom{-\frac{1}{2}}{n} \quad (4.21)$$

Remark 4.4 In the following identity, we evaluate any non-integral factorial value via the Gamma function, i.e. $\Gamma(x) = (x-1)!$ whenever x is not a negative integer.

$$\sum_{k=0}^n (-1)^k \frac{1}{k! \left(\frac{1}{2} - k\right)!} = \frac{2(-1)^n}{n! \left(-\frac{1}{2} - n\right)!} \quad (4.22)$$

$$\sum_{k=0}^n \binom{2k}{k} \frac{1}{2^{2k}(2k-1)} = \frac{-1}{2^{2n}} \binom{2n}{n} \quad (4.23)$$

With the $\frac{-1}{2}$ Transformation

$$\sum_{k=0}^n \binom{2k}{k} \frac{1}{2^{2k}} = \frac{2n+1}{2^{2n}} \binom{2n}{n} \quad (4.24)$$

Remark 4.5 The following two identities are the solution to Problem E995, p. 700 of the December 1951 American Mathematical Monthly.

$$\sum_{k=1}^n \binom{2k}{k} \frac{k}{2^{2k}} = \frac{n(2n+1)}{3 * 2^{2n}} \binom{2n}{n}, \quad n \geq 1 \quad (4.25)$$

$$\sum_{k=1}^n \binom{2k}{k} \frac{k}{2^{2k}} = \frac{n(n+1)}{3 * 2^{2n+1}} \binom{2n+2}{n+1}, \quad n \geq 1 \quad (4.26)$$

$$\sum_{k=1}^n \binom{2k}{k} \frac{k^2}{2^{2k}} = \frac{n(2n+1)(3n+2)}{3 * 5 * 2^{2n}} \binom{2n}{n}, \quad n \geq 1 \quad (4.27)$$

$$\sum_{k=1}^n \binom{2k}{k} \frac{k^3}{2^{2k}} = \frac{n(2n+1)(15n^2+18n+2)}{3 * 5 * 7 * 2^{2n}} \binom{2n}{n}, \quad n \geq 1 \quad (4.28)$$

Two Identities using the Vandermonde Convolution

Remark 4.6 In the following two identities, we assume α is a positive integer.

$$\sum_{\alpha=1}^n \binom{n+1}{\alpha} \binom{n-1}{\alpha-1} = \binom{2n}{n}, \quad n \geq 1 \quad (4.29)$$

$$\sum_{\alpha=1}^n \binom{n}{\alpha}^2 \frac{\alpha}{n-\alpha+1} = \binom{2n}{n+1}, \quad n \geq 1 \quad (4.30)$$

Sparre-Andersen Formulas

Remark 4.7 Let $f(x)$ be a given function and $F(k) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(j)$. Then,

$$\sum_{k=0}^n (-1)^k \binom{x}{k} F(k) = (-1)^n \binom{x-1}{n} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x}{x-j} f(j), \quad (4.31)$$

where x is an arbitrary real or complex number such that $x - j \neq 0$ for $0 \leq j \leq n$.

Remark 4.8 The following two identities are found in Erik Sparre-Andersen's "Two Summation Formulae for Product Sums of Binomial Coefficients", *Mathematica Scandinavica*, Vol. 1, 1953, pp. 261-262.

$$\begin{aligned} \sum_{k=0}^{\alpha} \binom{x}{k} \binom{-x}{n-k} &= \frac{\alpha-x}{n} \binom{x}{\alpha} \binom{-x-1}{n-\alpha-1} \\ &= \frac{n-\alpha}{n} \binom{x-1}{\alpha} \binom{-x}{n-\alpha} = \frac{-\alpha-1}{n} \binom{x}{\alpha+1} \binom{-x-1}{n-\alpha-1}, \end{aligned} \quad (4.32)$$

for all integers n, α such that $n \geq 1$ and $0 \leq \alpha \leq n$.

$$\sum_{k=0}^{\alpha} \binom{x}{k} \binom{1-x}{n-k} = \frac{(n-1)(1-x) - \alpha}{n(n-1)} \binom{x-1}{\alpha} \binom{-x}{n-\alpha-1}, \quad (4.33)$$

for all integers n, α such that $n \geq 2$ and $0 \leq \alpha \leq n-1$.

Remark 4.9 In the following two identities, assume x and z are arbitrary real or complex numbers, where $x - j \neq 0$ whenever $0 \leq j \leq \alpha$.

$$\sum_{k=0}^{\alpha} \binom{x}{k} \binom{z}{n-k} = (-1)^{\alpha} \binom{x-1}{\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \binom{z+j}{n} \frac{x}{x-j} \quad (4.34)$$

$$\sum_{k=0}^{\alpha} \binom{-x}{k} \binom{z}{n-k} = \binom{x+\alpha}{\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \binom{z+j}{n} \frac{x}{x+j} \quad (4.35)$$

$$\begin{aligned} \sum_{k=\alpha+1}^n \binom{x}{k} \binom{-x}{n-k} &= \sum_{k=0}^{\alpha} \binom{x}{k} \binom{-x}{n-k} \\ &= \frac{x-\alpha}{n} \binom{x}{\alpha} \binom{-x-1}{n-\alpha-1}, \quad n \geq 1, 0 \leq \alpha < n \end{aligned} \quad (4.36)$$

Saalschütz Formula

$$\sum_{k=a}^n (-1)^{k-a} \binom{n}{k} \frac{1}{k} = \sum_{k=a}^n \binom{k-1}{a-1} \frac{1}{k}, \quad 1 \leq a \leq n \quad (4.37)$$

Derivatives Involving Identity (4.19)

Remark 4.10 For the following three identities, we refer the reader to Version 1 of the Generalized Chain Rule provided in Section 3.6. Note that we let $D_x^n f(x)$ denote the n^{th} derivative of $f(x)$ with respect to x . Furthermore, we assume x , and z are real or complex numbers, while r is a real number.

$$D_x^n z^r = \sum_{k=0}^n (-1)^{n-k} \binom{r}{k} \binom{r-k-1}{n-k} z^{r-k} D_x^n z^k \quad (4.38)$$

$$D_x^n z^{-r} = \sum_{k=0}^n (-1)^k \binom{r+k-1}{k} \binom{r+n}{n-k} \frac{1}{z^{r+k}} D_x^n z^k \quad (4.39)$$

$$D_x^n z^{-r} = r \binom{r+n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{r+k} \frac{1}{z^{r+k}} D_x^n z^k \quad (4.40)$$

5 Reciprocal Pascal's Identity with Series

5.1 Additive Forms of Reciprocal Pascal's Identity

$$\frac{1}{\binom{n+1}{k}} + \frac{1}{\binom{n+1}{k+1}} = \frac{n+2}{n+1} \frac{1}{\binom{n}{k}} \quad (5.1)$$

$$\frac{1}{\binom{2n+1}{k}} + \frac{1}{\binom{2n+1}{k+1}} = \frac{2n+2}{2n+1} \frac{1}{\binom{2n}{k}} \quad (5.2)$$

5.1.1 Applications of Equation (5.2)

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{\binom{2n}{k}} = \frac{1}{n+1}, \quad n \geq 1 \quad (5.3)$$

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} k}{\binom{2n}{k}} = \frac{n}{n+1}, \quad n \geq 1 \quad (5.4)$$

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} (n-k)}{\binom{2n}{k}} = 0, \quad n \geq 1 \quad (5.5)$$

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}(n+k)}{\binom{2n}{k}} = \frac{2n}{n+1}, \quad n \geq 1 \quad (5.6)$$

$$\sum_{k=1}^n \frac{(-1)^{k-1}}{\binom{2n}{k}} = \frac{1}{2(n+1)} + \frac{(-1)^{n-1}}{2\binom{2n}{n}}, \quad n \geq 1 \quad (5.7)$$

5.2 Subtraction Form of Reciprocal Pascal's Identity

$$\frac{1}{\binom{n+1}{k}} - \frac{1}{\binom{n+1}{k+1}} = \frac{n-2k}{n+1} \frac{1}{\binom{n}{k}} \quad (5.8)$$

5.2.1 Applications of Equation (5.8)

$$\sum_{k=1}^r \frac{n-2k}{\binom{n}{k}} = 1 - \frac{n+1}{\binom{n+1}{r+1}}, \quad n, r \geq 1 \quad (5.9)$$

$$\sum_{k=1}^n \frac{2k-n}{\binom{n}{k}} = n, \quad n \geq 1 \quad (5.10)$$

$$\sum_{k=1}^{n-1} \frac{2k-n}{\binom{n}{k}} = 0, \quad n \geq 2 \quad (5.11)$$

$$\sum_{k=0}^n \frac{n-2k}{\binom{n}{k}} = 0 \quad (5.12)$$

$$\sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} = (1 + (-1)^n) \frac{n+1}{n+2} \quad (5.13)$$

5.3 Generalized Additive Form of Reciprocal Pascal's Identity

$$\frac{1}{\binom{x}{k}} = \frac{x+1}{x+2} \left(\frac{1}{\binom{x+1}{k}} + \frac{1}{\binom{x+1}{k+1}} \right), \quad x \neq -1, -2 \quad (5.14)$$

5.3.1 Applications of Equation (5.14)

$$\sum_{k=0}^n \frac{(-1)^k}{\binom{x}{k}} = \frac{x+1}{x+2} \left(1 + \frac{(-1)^n}{\binom{x+1}{n+1}} \right), \quad x \neq -1, -2 \quad (5.15)$$

$$\sum_{k=0}^n \frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{3} \left(1 + \frac{(2n+1)2^{2n+2}}{\binom{2n+2}{n+1}} \right) \quad (5.16)$$

$$\sum_{k=0}^n \frac{1}{\binom{x+k}{k}} = \frac{x}{x-1} \left(1 - \frac{1}{\binom{x+n}{n+1}} \right), \quad x \neq 0, 1 \quad (5.17)$$

$$\sum_{k=j}^n \frac{1}{\binom{k}{j}} = \begin{cases} \sum_{k=1}^n \frac{1}{k}, & j = 1 \\ \frac{j}{j-1} \left(1 - \frac{1}{\binom{n}{j-1}} \right), & j > 1 \end{cases} \quad (5.18)$$

$$\sum_{k=j}^{\infty} \frac{1}{\binom{k}{r}} = \frac{j}{(r-1)\binom{j}{r}}, \quad r > 1 \quad (5.19)$$

Combinatorial Identities: Table I: Intermediate Techniques for Summing Finite Series

From the seven unpublished manuscripts of H. W. Gould
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1 Coefficient Comparison

Remark 1.1 *Throughout this chapter, we will assume, unless otherwise specified, that x is an arbitrary complex number and n is a nonnegative integer.*

1.1 Expansions of $(1 + x)^n(a + x)^n$

Remark 1.2 *In this section, we will assume a is an arbitrary complex number.*

$$\sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (a-1)^k = \sum_{k=0}^n \binom{n}{k}^2 a^{n-k} \quad (1.1)$$

1.1.1 Equation (1.1) with $a = 0$

$$\sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (-1)^k = 1 \quad (1.2)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{n} = (-1)^n \quad (1.3)$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{n-k} \binom{2k}{k} = (-1)^n \quad (1.4)$$

$$\sum_{k=1}^n (-1)^{n-k} \binom{n+k-1}{k} \binom{n-1}{k-1} = 1, \quad n \geq 1 \quad (1.5)$$

1.1.2 Equation (1.1) with $a = 1$

Special Case of the Vandermonde Convolution

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad (1.6)$$

1.1.3 Equation (1.1) with $a = -1$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n-k}{2n} 2^k = (-1)^n \binom{2n}{n} \quad (1.7)$$

1.1.4 Equation (1.1) with $a = 2$

Remark 1.3 *The following Identity, due to Leo Moser, is Problem E799 of The American Math. Monthly, 1948, P. 30.*

$$\sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} = \sum_{k=0}^n \binom{n}{k}^2 2^{n-k} \quad (1.8)$$

1.1.5 Shifted Version of Equation (1.1)

$$\sum_{k=0}^{n+1} \binom{n}{k} \binom{2n-k}{n+1-k} (a-1)^k = \sum_{k=1}^n \binom{n}{k} \binom{n}{n+1-k} a^{n+1-k}, \quad n \geq 1 \quad (1.9)$$

Equation (1.9) with $a = 1$

$$\sum_{k=1}^n \binom{n}{k} \binom{n}{n+1-k} = \binom{2n}{n+1}, \quad n \geq 1 \quad (1.10)$$

Equation (1.9) with $a = 0$

$$\sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{2n-k}{n+1-k} = 0 \quad (1.11)$$

1.2 Expansions of $(\alpha x + \beta)^p(1 + x)^q$

Remark 1.4 In this subsection, we assume α and β are arbitrary real or complex numbers. We note that q is an arbitrary real number, while p is a nonnegative integer.

Remark 1.5 The following identity is found in Wilhelm Ljunggren's "Et elemnært bevis for en formel av A.C. Dixon", Norsk Matematisk Tidsskrift, 29.Ågang, 1947, pp. 35-38.

$$\sum_{k=0}^p \binom{p}{k} \binom{q}{k} \alpha^{p-k} \beta^k = \sum_{k=0}^p \binom{p}{k} \binom{q+k}{k} (\alpha - \beta)^{p-k} \beta^k \quad (1.12)$$

1.2.1 Dixon Sums

Dixon's Identity

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n} \binom{3n}{n} \quad (1.13)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n-k}{2n} \binom{2n+k}{2n} \quad (1.14)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{n+k}{n} \quad (1.15)$$

1.3 Expansions of $(1 + x)^{2n}(1 - x)^{2n}$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \quad (1.16)$$

$$\sum_{k=0}^n \binom{2n}{2k}^2 = \frac{1}{2} \binom{4n}{2n} + \frac{(-1)^n}{2} \binom{2n}{n} \quad (1.17)$$

$$\sum_{k=0}^{n-1} \binom{2n}{2k+1}^2 = \frac{1}{2} \binom{4n}{2n} - \frac{(-1)^n}{2} \binom{2n}{n}, \quad n \geq 1 \quad (1.18)$$

1.4 Expansions of $(1 - x)^n(1 + x)^n$

Remark 1.6 *In this section, we assume r is a nonnegative integer.*

$$\sum_{k=0}^r (-1)^k \binom{n}{k} \binom{n}{r-k} = \begin{cases} 0, & \text{if } r \text{ is odd} \\ (-1)^{\frac{r}{2}} \binom{n}{\frac{r}{2}}, & \text{if } r \text{ is even} \end{cases} \quad (1.19)$$

1.4.1 Restatements of Equation (1.19)

First Restatement of Equation (1.19)

$$\sum_{k=0}^{2r+1} (-1)^k \binom{n}{k} \binom{n}{2r-k+1} = 0 \quad (1.20)$$

$$\sum_{k=0}^{2r} (-1)^k \binom{n}{k} \binom{n}{2r-k} = (-1)^r \binom{n}{r} \quad (1.21)$$

Second Restatement of Equation (1.19)

$$\sum_{k=0}^r (-1)^k \binom{n}{k} \binom{n}{r-k} = (-1)^{\lfloor \frac{r}{2} \rfloor} \binom{n}{\lfloor \frac{r}{2} \rfloor} \frac{1 + (-1)^r}{2}, \quad (1.22)$$

where, for any real number x , we let $\lfloor x \rfloor$ denote the greatest integer in x .

1.4.2 Applications of Equation (1.21)

$$\sum_{k=0}^{2r} (-1)^k \binom{2r}{k} \binom{2n-2r}{n-k} = (-1)^r \frac{\binom{n}{r} \binom{2n}{n}}{\binom{2n}{2r}} \quad (1.23)$$

$$\sum_{k=0}^r \binom{n}{2k} \binom{n}{2r-2k} = \frac{1}{2} \binom{2n}{2r} + \frac{(-1)^r}{2} \binom{n}{r} \quad (1.24)$$

$$\sum_{k=0}^{r-1} \binom{n}{2k+1} \binom{n}{2r-2k-1} = \frac{1}{2} \binom{2n}{2r} - \frac{(-1)^r}{2} \binom{n}{r}, \quad r \geq 1 \quad (1.25)$$

$$\sum_{k=0}^r (-1)^k \binom{n}{k} \binom{n}{2r-k} = \frac{(-1)^r}{2} \left(\binom{n}{r} + \binom{n}{r}^2 \right) \quad (1.26)$$

$$\sum_{k=0}^r (-1)^k \binom{2r}{k} \binom{2n-2r}{n-k} = \frac{(-1)^r}{2} \left(\binom{n}{r} + \binom{n}{r}^2 \right) \frac{\binom{2n}{2r}}{\binom{2n}{2r}} \quad (1.27)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k}^2 = \frac{(-1)^n}{2} \left(\binom{2n}{n} + \binom{2n}{n}^2 \right) \quad (1.28)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{2k}^2 = \frac{1}{4} \binom{4n}{2n} + \frac{(-1)^n}{4} \binom{2n}{n} + \frac{1 + (-1)^n}{4} \binom{2n}{n}^2 \quad (1.29)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{2n}{2k+1}^2 = \frac{1}{4} \binom{4n}{2n} - \frac{(-1)^n}{4} \binom{2n}{n} + \frac{1 - (-1)^n}{4} \binom{2n}{n}^2, \quad n \geq 1 \quad (1.30)$$

$$\sum_{k=0}^n \binom{4n}{2k}^2 = \frac{1}{4} \binom{8n}{4n} + \frac{1}{4} \binom{4n}{2n} + \frac{1}{2} \binom{4n}{2n}^2 \quad (1.31)$$

$$\sum_{k=0}^n \binom{4n+2}{2k+1}^2 = \frac{1}{4} \binom{8n+4}{4n+2} + \frac{1}{4} \binom{4n+2}{2n+1} + \frac{1}{2} \binom{4n+2}{2n+1}^2 \quad (1.32)$$

1.4.3 Applications of Equation (1.19) to Expansions of $(1+x)^n(x-1)^n$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} 2^k = 0, \quad \text{if } n \text{ is odd} \quad (1.33)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n-k}{2n} 2^k = (-1)^n \binom{2n}{n} \quad (1.34)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} 2^k = (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1 + (-1)^n}{2} \quad (1.35)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{n} \frac{1}{2^k} = 0, \quad \text{if } n \text{ is odd} \quad (1.36)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+k}{2n} \frac{1}{2^k} = (-1)^n \binom{2n}{n} \frac{1}{2^{2n}} \quad (1.37)$$

1.5 Expansions of $(1-x)^m(1+x)^{2n-m}$

$$\sum_{j=0}^n (-1)^j \binom{x}{j} \binom{2n-x}{n-j} = (-1)^n \sum_{j=0}^n (-1)^j \binom{2n-j}{n-j} \binom{2n-x}{j} 2^j \quad (1.38)$$

$$\sum_{j=0}^n (-1)^j \binom{x}{j} \binom{2n-x}{n-j} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{x}{j} \binom{2n-2x}{n-2j} \quad (1.39)$$

$$\sum_{j=0}^n (-1)^j \binom{x}{j} \binom{2n-x}{n-j} = (-1)^n 2^{2n} \binom{\frac{x-1}{2}}{n} \quad (1.40)$$

$$\sum_{j=0}^n (-1)^j \binom{x}{j} \binom{2n-x}{n-j} = \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-x), \quad n \geq 1 \quad (1.41)$$

$$\sum_{j=0}^n (-1)^j \binom{x}{j} \binom{2n-x}{n-j} = \sum_{k=0}^n \binom{-n-1}{n-k} \binom{2n-x}{k} 2^k \quad (1.42)$$

1.5.1 Applications of Equation (1.40)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{2k} \binom{2n-x}{n-2k} = \frac{1}{2} \left(\binom{2n}{n} + (-1)^n 2^{2n} \binom{\frac{x-1}{2}}{n} \right) \quad (1.43)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{x}{2k+1} \binom{2n-x}{n-2k-1} = \frac{1}{2} \left(\binom{2n}{n} - (-1)^n 2^{2n} \binom{\frac{x-1}{2}}{n} \right), \quad n \geq 1 \quad (1.44)$$

1.5.2 Special Cases of Equation (1.40)

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 = (-1)^n 2^{2n} \binom{\frac{n-1}{2}}{n} = (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1 + (-1)^n}{2} \quad (1.45)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{2k} = 2^{4n} \binom{\frac{n-1}{2}}{2n} \quad (1.46)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{2k} = \frac{2^{4n}}{(2n)!} \prod_{k=0}^{2n-1} (2k+1-n), \quad n \geq 1 \quad (1.47)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{2k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{3n}{2n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{3n}{n+k} \quad (1.48)$$

$$\sum_{k=0}^n (-1)^k \binom{n - \frac{1}{2}}{k} \binom{n + \frac{1}{2}}{n-k} = (-1)^n 2^{2n} \binom{\frac{2n-3}{4}}{n} \quad (1.49)$$

$$\sum_{k=0}^n (-1)^k \binom{n - \frac{1}{2}}{k} \binom{n + \frac{1}{2}}{n-k} = \frac{\binom{2n}{n}^2 (2n+1)}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \frac{1}{\binom{2k}{k} \binom{2n-2k}{n-k} (2k+1)} \quad (1.50)$$

Remark 1.7 In the following identity, we evaluate $(\frac{1}{2})!$ by the Gamma function, namely, $(\frac{1}{2})! = \Gamma(\frac{3}{2})$.

$$\sum_{k=0}^n (-1)^k \binom{n - \frac{1}{2}}{k} \binom{n + \frac{1}{2}}{n-k} = \frac{\pi^2 (2n+1) \binom{2n}{n}^2}{2^{4n+2}} \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \binom{k}{\frac{1}{2}} \binom{n-k}{\frac{1}{2}} \frac{1}{2k+1} \quad (1.51)$$

$$\sum_{k=0}^n (-1)^k \binom{\frac{-1}{2}}{k} \binom{2n + \frac{1}{2}}{n-k} = \frac{2^n}{n!} \prod_{k=0}^{n-1} \left(2k + \frac{3}{2}\right), \quad n \geq 1 \quad (1.52)$$

$$\sum_{k=0}^n \frac{\binom{2n}{n+k} \binom{2k}{k}}{\binom{2n+2k}{n+k} (2n+2k+1)} = \frac{2^{3n}}{(4n+1) \binom{4n}{2n} n!} \prod_{k=0}^{n-1} \left(2k + \frac{3}{2}\right), \quad n \geq 1 \quad (1.53)$$

Remark 1.8 *The following two identities are found in C. van Ebbenhorst Tengbergen, “Über die Identitäten...etc.” Nieuw. Arch. Wiskde., Vol. 18 (1943), pp. 1-7. We assume a is a nonnegative integer.*

$$\sum_{k=a}^n (-1)^k \binom{-2a}{k-a} \binom{2n}{n-k} = (-1)^a \frac{\binom{2n}{n}}{\binom{2a}{a}} \binom{n}{a}, \quad n \geq a \geq 0 \quad (1.54)$$

$$\sum_{k=0}^n (-1)^k \binom{-2a}{k} \binom{2n+2a}{n-k} = \frac{\binom{2n+2a}{n+a} \binom{n+a}{a}}{\binom{2a}{a}}, \quad (1.55)$$

1.6 Expansions of $(1+x)^z(1-x)^{-z}$

Remark 1.9 *In the following section, we assume z is an arbitrary complex number.*

Mitlag-Leffler's Polynomials

$$\sum_{j=0}^k (-1)^j \binom{-z}{j} \binom{-z}{k-j} = \sum_{j=0}^{k-1} \binom{z}{j+1} \binom{k-1}{j} 2^{j+1}, \quad k \geq 1 \quad (1.56)$$

1.7 Expansions of $(1+z^p)^x(1+z)^{n-y-px}$

Remark 1.10 *In the following section, we assume z and y are arbitrary complex numbers, while p is a positive integer. A reference for these expansions is “Another Note on the Hermite Polynomials” by M. P. Drazin, *American Math. Monthly*, Vol. 64, 1957, pp. 89-91.*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{y-2x}{n-2k} = \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{y-2k}{n-k} 2^k \quad (1.57)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{x}{n-2k} = \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{3x-2k}{n-k} 2^k \quad (1.58)$$

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{x}{k} \binom{y-3x}{n-3k} = \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{y-2k}{n-k} 3^k \quad (1.59)$$

$$\begin{aligned}
(-1)^n \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{3x-2k}{n-k} 3^k &= \binom{x}{\frac{n}{3}} \frac{(-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+2}{3} \rfloor}}{2} \\
&= \binom{x}{\frac{n}{3}} \frac{(-1)^{\lfloor \frac{n}{3} \rfloor} - (-1)^{\lfloor \frac{n-1}{3} \rfloor}}{2}
\end{aligned} \tag{1.60}$$

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{x}{k} \binom{x}{n-3k} = \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{4x-2k}{n-k} 3^k \tag{1.61}$$

1.8 Expansions of $(x^2 + 2x)^n$

Remark 1.11 *In the following section, we will assume k is a nonnegative integer, and y is an arbitrary complex number.*

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{2j}{k} = (-1)^n \binom{n}{k-n} 2^{2n-k} \tag{1.62}$$

1.8.1 Applications of Equation (1.62)

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{2j}{n} = (-1)^n 2^n \tag{1.63}$$

$$\sum_{j=0}^n \binom{n}{j} \binom{j}{k-j} 2^{2j} = 2^k \binom{2n}{k} \tag{1.64}$$

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{2j}{j} \frac{1}{2^{2j}} = \frac{1}{2^n} \binom{2n}{n} \tag{1.65}$$

$$\sum_{j=0}^n \binom{n}{j} \binom{n-j}{k-2j} 2^{k-2j} = \binom{2n}{k} \tag{1.66}$$

1.8.2 Generalizations of Equation (1.64)

$$\sum_{k=0}^n \binom{x}{k} \binom{k}{n-k} 2^{2k} = 2^n \binom{2x}{n} \quad (1.67)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{x-k}{n-2k} \frac{1}{2^{2k}} = 2^{-n} \binom{2x}{n} \quad (1.68)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{x-k}{n-2k} 2^{n-2k} = \binom{2x}{n} \quad (1.69)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{y+k}{n-k} 2^{2k} = \sum_{k=0}^n \binom{2x}{k} \binom{y}{n-k} 2^k \quad (1.70)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} 2^k = \sum_{k=0}^n \binom{x}{k} \binom{x+y-k}{n-k} \quad (1.71)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{y+k}{n-k} 2^{2k} = \sum_{k=0}^n \binom{2x}{k} \binom{2x+y-k}{n-k} \quad (1.72)$$

Remark 1.12 *The following two identities are associated with Problem 3803 [1936, P.580; 1938, pp. 633-634] in The American Math. Monthly*

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{2x-2k}{n-k} 2^{2k} = (-1)^n \binom{2x}{n} \quad (1.73)$$

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{4x-2k}{n-k} 2^{2k} = (-1)^{\frac{n}{2}} \frac{1 + (-1)^n}{2} \binom{2x}{\frac{n}{2}} \quad (1.74)$$

1.9 Expansions by I. J. Schwatt

Remark 1.13 *In the following section, we assume z is an arbitrary complex number. We also let $((z^n))$ denote the coefficient of z^n in the power series expansion. For a reference, see “Introduction to the Operations with Series” by I. J. Schwatt, Univ. of Pa. Press, 1924, pp. 93-94. A reprint of this book was issued in 1962 by Chelsea Publ. Co., New York, N. Y.*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{x-k}{n-2k} 2^{n-2k} = ((z^n))(1+z)^{2x} \quad (1.75)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{x-k}{n-2k} \frac{1}{2^{n-2k}} = \frac{1}{2^x} ((z^n))(2z^2 + z + 2)^x \quad (1.76)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{k} \binom{x-k}{n-2k} 3^{n-2k} = ((z^n))(z^2 + 3z + 1)^x \quad (1.77)$$

1.10 Coefficient Comparisons Involving the Companion Binomial Theorem

Remark 1.14 *In this section, we will assume, unless otherwise specified, that a and r are arbitrary real or complex numbers.*

$$\sum_{k=0}^n \binom{a+k}{k} \binom{r+n-k}{n-k} = \binom{a+r+n+1}{n} \quad (1.78)$$

$$\sum_{k=0}^n \binom{r+k}{k} \binom{r+n-k}{n-k} = \binom{2r+1+n}{n} \quad (1.79)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{r+k}{k} \binom{r+n-k}{n-k} = \frac{1}{2} \binom{2r+n+1}{n} - \frac{1+(-1)^n}{4} \binom{r+\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}, \quad n \geq 1 \quad (1.80)$$

$$\sum_{k=0}^n \binom{r+k}{k} \binom{r+2n-k}{2n-k} = \frac{1}{2} \binom{2r+2n+1}{2n} + \frac{1}{2} \binom{r+n}{n}^2 \quad (1.81)$$

$$\sum_{k=0}^n \binom{r+k}{k} \binom{r+2n-k+1}{2n-k+1} = \frac{1}{2} \binom{2r+2n+2}{2n+1} \quad (1.82)$$

$$\sum_{i=0}^k (-1)^i \binom{n}{i} \binom{n+k-i}{n} = 1 \quad (1.83)$$

2 Linear Algebra Techniques Applied to Series

Remark 2.1 Throughout this chapter, we assume, unless otherwise specified, that n is a nonnegative integer. We let x and z denote arbitrary real or complex numbers. If x is a real number, we let $[x]$ denote the integer part of x .

2.1 Series Derived from M^n

Remark 2.2 Throughout this section, we assume M is a 2×2 matrix of complex entries with two distinct eigenvalues, t_1 and t_2 .

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i (t_1 t_2)^i (t_1 + t_2)^{n-2i-1} \binom{n-i-1}{i} = \frac{t_2^n - t_1^n}{t_2 - t_1} \quad (2.1)$$

2.1.1 Applications of Equation (2.1)

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left(\frac{t}{(1+t)^2} \right)^i \binom{n-i}{i} = \frac{t^{n+1} - 1}{(t-1)(1+t)^n}, \quad t \neq 1 \quad (2.2)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} z^i = \frac{(2z + 1 + \sqrt{1+4z})^{n+1} + (-1)^n (2z)^{n+1}}{2^{\frac{n}{2}} (2z + 1 + \sqrt{1+4z})^{\frac{n}{2}} (4z + 1 + \sqrt{1+4z})} \quad (2.3)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} \frac{1}{2^{2i}} = \frac{n+1}{2^n} \quad (2.4)$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{n-k} 2^{2k} = (-1)^n (2n+1) \quad (2.5)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 2^i = \frac{2^{n+1} + (-1)^n}{3} \quad (2.6)$$

$$\sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{2^k} = \sum_{k=0}^n \binom{n+k}{2k} \frac{1}{2^k} = \frac{2^{2n+1} + 1}{3 * 2^n} \quad (2.7)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} = F_n, \quad (2.8)$$

where F_n is the n^{th} Fibonacci number given by the recurrence $F_n = F_{n-1} + F_{n-2}$, $F_0 = 1$, $F_1 = 1$.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} = \frac{(-1+i\sqrt{3})^{n+1} - 2^{n+1}}{2^{\frac{n}{2}}(-1+i\sqrt{3})^{\frac{n}{2}}(-3+i\sqrt{3})}, \quad \text{where } i = \sqrt{-1} \quad (2.9)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} = \frac{(-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+1}{3} \rfloor}}{2}, \quad n = 3m + a, \quad (a = 0, 1, 2, \dots) \quad (2.10)$$

Limiting Case of Equation (2.1)

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \binom{n-i-1}{i} \frac{1}{2^{2i}} = \frac{n}{2^{n-1}}, \quad n \geq 1 \quad (2.11)$$

$$\sum_{i=0}^n (-1)^i (t_1 t_2)^{n-i} (t_1 + t_2)^{2i} \binom{n+i}{2i} = (-1)^n \frac{t_2^{2n+1} - t_1^{2n+1}}{t_2 - t_1} \quad (2.12)$$

$$\sum_{k=0}^n \binom{n+k}{2k} (t_1 - t_2)^{2k} (t_1 t_2)^{n-k} = \frac{t_1^{2n+1} + t_2^{2n+1}}{t_1 + t_2} \quad (2.13)$$

$$\sum_{i=0}^n (-1)^i (t_1 t_2)^{n-i} (t_1 + t_2)^{2i+1} \binom{n+i+1}{2i+1} = (-1)^n \frac{t_2^{2n+2} - t_1^{2n+2}}{t_2 - t_1} \quad (2.14)$$

$$\sum_{i=0}^n (-1)^i (t_1 t_2)^{n-i} (t_1 + t_2)^{2i+1} \binom{n+i}{2i+1} = (-1)^{n-1} \frac{t_1 t_2 (t_2^{2n} - t_1^{2n})}{t_2 - t_1} \quad (2.15)$$

$$\sum_{k=0}^n \binom{n+k}{2k+1} (t_1 - t_2)^{2k+1} (t_1 t_2)^{n-k-1} = \frac{t_1^{2n} - t_2^{2n}}{t_1 + t_2} \quad (2.16)$$

$$\sum_{i=0}^n (-1)^i (t_1 t_2)^{n-i} (t_1 + t_2)^{2i+1} \left(\binom{n+i+1}{2i+1} + \binom{n+i}{2i+1} \right) = (-1)^n (t_2^{2n+1} + t_1^{2n+1}) \quad (2.17)$$

Remark 2.3 A reference for Equation (2.18) is the Solution to Problem 4356 [1949, 479] in *American Math. Monthly*, April 1952, p. 268.

$$\sum_{i=0}^n (-1)^i (t_1 t_2)^{n-i} (t_1 + t_2)^{2i+1} \binom{n+i}{2i} \frac{1}{2i+1} = (-1)^n \frac{t_2^{2n+1} + t_1^{2n+1}}{2n+1} \quad (2.18)$$

Limiting Case of Equation (2.18)

$$\sum_{i=0}^n (-1)^i \binom{n+i}{2i} \frac{2^{2i}}{2i+1} = \frac{(-1)^n}{2n+1} \quad (2.19)$$

$$\sum_{i=0}^n (-1)^i (t_1 t_2)^{n-i} (t_1 + t_2)^{2i-1} \binom{n+i-1}{2i-1} = (-1)^n \frac{t_2^{2n} - t_1^{2n}}{t_2 - t_1} \quad (2.20)$$

2.2 Convolutions and Determinants

Remark 2.4 For Equation (2.22), we give the following system of equations

$$a_k = \sum_{j=0}^k b_j c_{k-j}, \quad (2.21)$$

where k is a nonnegative integer, and b_j is independent of k . We further assume that the a_k and b_j are known sequences (with $b_0 \neq 0$). Our goal is to solve for c_n . This is done in Equation (2.22). Note the right side of Equation (2.22) involves the determinant of an $n \times n$ matrix.

$$c_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} a_n & b_1 & b_2 & b_3 & \dots & b_n \\ a_{n-1} & b_0 & b_1 & b_2 & \dots & b_{n-1} \\ a_{n-2} & 0 & b_0 & b_1 & \dots & b_{n-2} \\ a_{n-3} & 0 & 0 & b_0 & \dots & b_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & 0 & 0 & 0 & \dots & b_0 \end{vmatrix} \quad (2.22)$$

Remark 2.5 For Equation (2.24), we are given the following system of equations

$$a_k = \sum_{j=0}^k b_j^k c_{k-j}, \quad (2.23)$$

where k is a nonnegative integer, and b_j^k depends on k . Please note that the superscript does not denote a power of b , but only the dependence on k as well as j . We further assume that a_k and b_j^k are known (with $b_0^k \neq 0$). Our goal is to solve for c_n . This is done in Equation (2.24). Note the right side of Equation (2.24) involves the determinant of an $n \times n$ matrix.

$$c_n = \frac{1}{\prod_{i=0}^n b_0^i} \begin{vmatrix} a_n & b_1^n & b_2^n & b_3^n & \dots & b_n^n \\ a_{n-1} & b_0^{n-1} & b_1^{n-1} & b_2^{n-1} & \dots & b_{n-1}^{n-1} \\ a_{n-2} & 0 & b_0^{n-2} & b_1^{n-2} & \dots & b_{n-2}^{n-2} \\ a_{n-3} & 0 & 0 & b_0^{n-3} & \dots & b_{n-3}^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & 0 & 0 & 0 & \dots & b_0^0 \end{vmatrix} \quad (2.24)$$

2.2.1 Application of Equation (2.24)

$$\sum_{j=0}^n (-1)^j \binom{n}{j} a_{n-j} = \begin{vmatrix} a_n & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n} \\ a_{n-1} & \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \dots & \binom{n-1}{n-1} \\ a_{n-2} & 0 & \binom{n-2}{0} & \binom{n-2}{1} & \dots & \binom{n-2}{n-2} \\ a_{n-3} & 0 & 0 & \binom{n-3}{0} & \dots & \binom{n-3}{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & 0 & 0 & 0 & \dots & b_0 \end{vmatrix} \quad (2.25)$$

2.3 Application of Equation (2.24) to Reciprocal Expansions

Remark 2.6 For Equation (2.27), we are given a power series representation for $f(x)$, namely, $f(x) = \sum_{i=0}^{\infty} a_i x^i$, with $a_0 \neq 0$. We assume α is an integer. Our goal is to calculate

$$\frac{1}{(f(x))^\alpha} = \sum_{j=0}^{\infty} b_j^\alpha x^j, \quad (2.26)$$

where the superscripts on the b 's are not exponents, but just an indicator of the entry in an array of numbers (See Remark 2.5). These calculations are done in Equation (2.27). Note the right side of Equation (2.27) is the determinant of an $n \times n$ matrix.

$$b_0^\alpha = \frac{1}{a_0^\alpha}, \quad \text{where the } \alpha \text{ subscript on } a_0 \text{ is an exponent,} \quad (2.27)$$

and if $n \geq 1$,

$$b_n^\alpha = \frac{1}{a_0^{n+\alpha} n!} \begin{vmatrix} 0 & (\alpha + n - 1)a_1 & (2\alpha + n - 2)a_2 & \dots & \dots & \alpha n a_n \\ 0 & (n - 1)a_0 & (\alpha + n - 2)a_1 & \dots & \dots & \alpha(n - 1)a_{n-1} \\ 0 & 0 & (n - 2)a_0 & \dots & \dots & \alpha(n - 2)a_{n-2} \\ 0 & 0 & 0 & (n - 3)a_0 & \dots & \alpha(n - 3)a_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & \dots & 1 \end{vmatrix}$$

2.3.1 An Alternative Form of Equation (2.27)

$$b_n^\alpha = \frac{(-1)^n}{a_0^{n+\alpha} n!} D,$$

where D is the determinant of the following $(n - 1) \times (n - 1)$ matrix

$$D = \begin{vmatrix} \alpha a_1 & a_0 & 0 & 0 & \dots & 0 \\ 2\alpha a_2 & (\alpha + 1)a_1 & 2a_0 & 0 & \dots & 0 \\ 3\alpha a_3 & (2\alpha + 1)a_2 & (\alpha + 2)a_1 & 3a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n\alpha a_n & ((n - 1)\alpha + 1)a_{n-1} & ((n - 2)\alpha + 2)a_{n-2} & ((n - 3)\alpha + 3)a_{n-3} & \dots & (\alpha + n - 1)a_1 \end{vmatrix}$$

2.4 Generalized Cauchy Convolution

Remark 2.7 For Equation (2.29), we are given a power series $\sum_{i=0}^{\infty} a_i x^i$. Our goal is to calculate product of this power series, namely

$$\left(\sum_{i=0}^{\infty} a_i x^i \right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{\alpha} x^n, \quad (2.28)$$

where α is an integer, $a_0 \neq 0$ if α is a negative integer, and the superscript on the B is not an exponent (see Remark 2.5). Equation (2.29) provides a solution for B_n^{α} . Note $B_0^{\alpha} = a_0^{\alpha}$, while if $n \geq 1$,

$$B_n^{\alpha} = \sum_{k=1}^n \binom{\alpha}{k} a_0^{\alpha-k} f_k(n), \quad (2.29)$$

where the superscript on a_0 denotes a power, and

$$f_k(n) = \sum_{\substack{\sum_{i=1}^k j_i = n \\ 1 \leq j_i \leq n}} a_{j_1} a_{j_2} a_{j_3} \dots a_{j_k}. \quad (2.30)$$

2.5 Inverse Function Expansions

Remark 2.8 Suppose we are given a power series $y = \sum_{i=1}^{\infty} a_i x^i$, with $a_1 \neq 0$. Our goal is to invert this series, namely to find $x = \sum_{\alpha=1}^{\infty} b_{\alpha} y^{\alpha}$, where $b_1 = \frac{1}{f_1(1)} = \frac{1}{a_1}$, and for $n \geq 1$,

$$b_n = \frac{(-1)^{n-1}}{a_1 \frac{n(n+1)}{2}} \begin{vmatrix} f_1(2) & f_2(2) & 0 & 0 & \dots & 0 \\ f_1(3) & f_2(3) & f_3(3) & 0 & \dots & 0 \\ f_1(4) & f_2(4) & f_3(4) & f_4(4) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ f_1(n) & f_2(n) & f_3(n) & f_4(n) & \dots & f_{n-1}(n) \end{vmatrix}, \quad (2.31)$$

with $f_k(n)$ defined by Equation (2.30).

3 Annihilation Coefficients

Remark 3.1 *In this chapter, we work with the series*

$$\sum_{k=1}^n c_k (x - k)^{n-k+1}, \quad (3.1)$$

where x is any real or complex number for which Equation (3.1) is defined. Our goal is to determine the c_k such that all terms in the sum, except x^n and the constant term, vanish simultaneously. All the formulas in this chapter are derived by this annihilation procedure. Also, we assume n is a positive integer, and m is a nonnegative integer.

$$\sum_{r=0}^n \binom{n}{r} (x - r)^{n-r} (r + 1)^{r-1} = (x + 1)^n \quad (3.2)$$

$$\sum_{r=0}^n (-1)^r \binom{n}{r} (x + r)^{n-r} (r + 1)^{r-1} = (x - 1)^n \quad (3.3)$$

$$\sum_{r=0}^n \binom{n}{r} (r + 1)^{r-1} (-r)^{n-r-1} = 1 - \frac{1}{n} \quad (3.4)$$

$$\sum_{r=0}^m \binom{n}{r} \binom{n-r}{m} (r + 1)^{r-1} (x - r)^{n-r-m} = \binom{n}{m} (x + 1)^{n-m} \quad (3.5)$$

$$\sum_{k=1}^n \binom{n}{k} k^k (n - k + 1)^{n-k-1} = n(n + 1)^{n-1} \quad (3.6)$$

4 Kummer Series Transformation

Remark 4.1 In this chapter, we start with the series $\sum_{k=a}^n f(k)$, where both n and a are nonnegative integers. For an arbitrary function $\varphi(n)$, we have

$$\sum_{k=a}^n f(k) = \sum_{k=a}^n \left(1 - \frac{f(n)\varphi(k)}{\varphi(n)f(k)}\right) f(k) + \frac{f(n)}{\varphi(n)} \sum_{k=a}^n \varphi(k) \quad (4.1)$$

Suppose that

- i. $\lim_{n \rightarrow \infty} \frac{f(n)}{\varphi(n)} = K \neq 0$
- ii. $S = \sum_{k=a}^{\infty} f(k)$ converges
- iii. $C = \sum_{k=a}^{\infty} \varphi(k)$ converges.

Then, Equation (4.1) implies

$$S = \sum_{k=a}^{\infty} \left(1 - K \frac{\varphi(k)}{f(k)}\right) f(k) + KC. \quad (4.2)$$

Equation (4.2) is known as the Kummer Series Transformation. A reference for this material is Konrad Knopp's "Theorie und Anwendung der unendlichen Reihen" Fourth Edition, Berlin, 1947. See Chapter VIII, P. 269.

4.1 Applications of the Kummer Series Transformation

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + 6 \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)(k+3)} \quad (4.3)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^r \frac{1}{k^2} + r! \sum_{k=1}^{\infty} \frac{(k-1)!}{k(k+r)!}, \quad r \geq 1 \quad (4.4)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r}{k} k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^r \frac{1}{k^2}, \quad r \geq 1 \quad (4.5)$$

Remark 4.2 In the following identity, we assume r is a positive integer.

$$\sum_{k=1}^{\infty} \frac{1}{k^r} = \sum_{k=1}^{\infty} \frac{(r+k)! - k^r k!}{(r+k)! k^r} + \frac{1}{(r-1)r!} \quad (4.6)$$

4.1.1 Knopp Applications of the Kummer Series Transformation

Remark 4.3 In section, we assume p is a positive integer, and α is arbitrary real or complex number, excluding the set of nonpositive integers. Define

$$S_p(\alpha) = \sum_{k=0}^{\infty} \left(\frac{1}{(k+\alpha)(k+\alpha+1)\dots(k+\alpha+p-1)} \right)^2. \quad (4.7)$$

Then,

$$S_p(\alpha) = \frac{(\alpha + \frac{3}{2}p - 1)}{(2p-1)(\alpha(\alpha+1)\dots(\alpha+p-1))^2} + \frac{p^3}{2(2p-1)} S_{p+1}(\alpha) \quad (4.8)$$

$$S_{n+1}(1) = \frac{2(2n-1)}{n^3} S_n - \frac{3}{n^2 n!^2} \quad (4.9)$$

$$S_{n+1}(1) = \frac{\pi^2}{6n!^2} \binom{2n}{n} - \frac{3}{n!^2} \binom{2n}{n} \sum_{j=1}^n \frac{1}{\binom{2j}{j} j^2} \quad (4.10)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}^2 k^2} = \frac{\pi^2}{6} \binom{2n}{n} - 3 \binom{2n}{n} \sum_{j=1}^n \frac{1}{\binom{2j}{j} j^2} \quad (4.11)$$

Limiting Case of Equation (4.11)

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k} k^2} = \frac{\pi^2}{18} \quad (4.12)$$

5 Series From Logarithmic Differentiation

Remark 5.1 Throughout this chapter, we will assume x is an arbitrary real or complex number, while n is a nonnegative integer.

5.1 Basic Logarithmic Differentiation Formula

Assume a is a nonnegative integer. If

$$F_n(x) = \prod_{k=a}^n (1 + u_k(x)), \quad (5.1)$$

then

$$\frac{F'_n(x)}{F_n(x)} = \sum_{k=a}^n \frac{u'_k(x)}{1 + u_k(x)}, \quad (5.2)$$

where $F'_n(x)$ is the derivative of $F_n(x)$ with respect to x .

5.2 Applications of the Basic Formula

$$\sum_{k=0}^n \frac{2^k x^{2^k}}{1 + x^{2^k}} = \frac{x}{1 - x} - \frac{2^{n+1} x^{2^{n+1}}}{1 - x^{2^{n+1}}}, \quad x \neq 1 \quad (5.3)$$

$$\sum_{k=0}^{\infty} \frac{2^k x^{2^k}}{1 + x^{2^k}} = \frac{x}{1 - x}, \quad |x| < 1 \quad (5.4)$$

$$\sum_{k=0}^{n-1} \frac{2^k}{1 + x^{2^k}} = \frac{1}{x - 1} + \frac{2^n}{1 - x^{2^n}}, \quad x \neq 1, \quad n \geq 1 \quad (5.5)$$

$$\sum_{k=0}^{\infty} \frac{2^k}{1 + x^{2^k}} = \frac{1}{x - 1}, \quad |x| > 1 \quad (5.6)$$

$$\sum_{k=0}^{n-1} \frac{x^{2^k} x^{2^k}}{(1 + x^{2^k})^2} = \frac{x}{(x - 1)^2} - \frac{2^{2n} x^{2^n}}{(1 - x^{2^n})^2}, \quad x \neq 1, \quad n \geq 1 \quad (5.7)$$

Remark 5.2 The following identity is Problem 728 of *Math. Magazine*, Vol. 42, May 1969, P. 153

$$\sum_{k=0}^{\infty} \frac{2^k x^{2^k} - 2^{k+1}}{x^{2^{k+1}} - x^{2^k} + 1} = \frac{x+2}{x^2+x+1}, \quad |x| > 1 \quad (5.8)$$

Remark 5.3 In the following two identities, we assume r is a positive integer.

$$\sum_{k=0}^{n-1} r^k \frac{\sum_{i=1}^{r-1} i x^{ir^k}}{\sum_{i=0}^{r-1} x^{ir^k}} = \frac{x}{1-x} - \frac{r^n x^{r^n}}{1-x^{r^n}}, \quad x \neq 1, \quad r \geq 2 \quad (5.9)$$

$$\sum_{k=0}^{\infty} 3^k \frac{x^{3^k} + 2x^{2*3^k}}{1+x^{3^k}+x^{2*3^k}} = \frac{x}{1-x}, \quad |x| < 1 \quad (5.10)$$

$$\sum_{k=1}^n \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{2^n} \cot\left(\frac{x}{2^n}\right) - \cot x \quad (5.11)$$

6 Vandermonde Convolution

6.1 Integral Version of the Vandermonde Convolution

Assume r and q are nonnegative integers. Then,

$$\sum_{i=0}^k \binom{r}{i} \binom{q}{k-i} = \binom{r+q}{k}. \quad (6.1)$$

Equation (6.1) is known as the (integral) Vandermonde Convolution.

6.1.1 Applications of Equation (6.1)

Remark 6.1 Throughout this section, we will assume r , q , and n are nonnegative integers. Also recall that for real x , $[x]$ is the integer part of x .

$$\sum_{i=0}^n \binom{n}{i} \binom{r+q-n}{r-i} = \binom{r+q}{r} \quad (6.2)$$

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad (6.3)$$

$$\sum_{k=0}^n \binom{n}{k}^2 k = \frac{n}{2} \sum_{k=0}^n \binom{n}{k}^2 = \frac{n}{2} \binom{2n}{n} \quad (6.4)$$

$$\sum_{k=0}^n \binom{n}{k}^2 k^2 = n^2 \binom{2n-2}{n-1} \quad (6.5)$$

$$\sum_{k=0}^n \binom{n}{k}^2 k^3 = \frac{n^2(n+1)}{2} \binom{2n-2}{n-1} \quad (6.6)$$

Remark 6.2 The following two identities are from Problem 3414, proposed by B. C. Wong, in *The American Math. Monthly*, March 1930, Vol. 37, No. 3.

$$\sum_{k=0}^n \binom{n}{k}^2 (n-2k)^2 = 2n \binom{2n-2}{n-1} \quad (6.7)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{k}^2 (n-2k)^2 = n \binom{2n-2}{n-1}, \quad n \geq 1 \quad (6.8)$$

6.2 Generalized Version of the Vandermonde Convolution

Assume x and y are arbitrary real or complex numbers. Assume n is a nonnegative integer. Then,

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}. \quad (6.9)$$

Equation (6.9) is the (generalized) Vandermonde Convolution.

6.2.1 Specific Evaluations of Equation (6.9) Using the $\frac{-1}{2}$ Transformation

$$\sum_{k=0}^n \binom{2n-2k}{n-k} \binom{2k}{k} = 2^{2n} \quad (6.10)$$

$$\sum_{k=0}^n \binom{n}{k}^2 \frac{1}{\binom{2n}{2k}} = \frac{2^{2n}}{\binom{2n}{n}} \quad (6.11)$$

$$\sum_{k=0}^n \binom{2n-2k}{n-k} \binom{2k}{k} \frac{1}{2k-1} = \begin{cases} 0, & n \geq 1 \\ -1, & n = 0 \end{cases} \quad (6.12)$$

$$\sum_{k=0}^n \binom{n}{k}^2 \frac{1}{\binom{2n}{2k} (2k-1)} = \begin{cases} 0, & n \geq 1 \\ -1, & n = 0 \end{cases} \quad (6.13)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{(2k-1)(2n-2k-1)} = \begin{cases} 1, & n = 0 \\ -4, & n = 1 \\ 0, & n \geq 2 \end{cases} \quad (6.14)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{2^{2k}} = \frac{1}{2^{2n}} \binom{2n}{n} \quad (6.15)$$

$$\sum_{k=0}^n (-1)^{k+1} \binom{n}{k} \binom{2k}{k} \frac{2k+1}{2^{2k}} = \frac{1}{2^{2n}(2n-1)} \binom{2n}{n} \quad (6.16)$$

6.2.2 Various Applications of Equation (6.9)

Remark 6.3 For the rest of Chapter 6, we assume, unless otherwise specified, that α and r are nonnegative integers. We also assume x , y , and z are arbitrary real or complex numbers.

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} \binom{k}{\alpha} = \binom{x}{\alpha} \binom{y+x-\alpha}{n-\alpha} \quad (6.17)$$

Remark 6.4 In the following two identities, we assume $f(x)$ is a polynomial of degree n , namely $f(x) = \sum_{i=0}^n a_i x^i = \sum_{j=0}^n b_j \binom{x}{j}$.

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} f(k) = \sum_{j=0}^n b_j \binom{x}{j} \binom{x+y-j}{n-j} \quad (6.18)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} f(k+z) = \sum_{j=0}^n b_j \sum_{\alpha=0}^j \binom{z}{j-\alpha} \binom{x}{\alpha} \binom{y+x-\alpha}{n-\alpha} \quad (6.19)$$

$$\sum_{k=0}^n \binom{n}{\alpha} \binom{x+\alpha}{n} = \sum_{j=0}^n \binom{n}{j} \binom{x}{j} 2^j \quad (6.20)$$

Remark 6.5 In the following identity, due to Laplace, assume u is a real or complex number, $u \neq -1$.

$$\sum_{k=0}^n \binom{x-n+k}{k} \frac{u^k}{(1+u)^k} = \frac{1}{(1+u)^n} \sum_{j=0}^n \binom{x+1}{j} u^j \quad (6.21)$$

$$\sum_{k=0}^n \binom{n+k}{k} ((1-x)^{n+1} x^k + x^{n+1} (1-x)^k) = 1, \quad x \neq 0, 1 \quad (6.22)$$

$$\sum_{k=0}^n \binom{n+k}{k} \frac{(x-1)^{n+1} - (x-1)^k}{x^k} = x^{n+1}, \quad x \neq 0, 1 \quad (6.23)$$

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{2n-k}{n} = \binom{2n-x}{n} \quad (6.24)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-k}{n} 2^{-2k} = \binom{2n+\frac{1}{2}}{n} = \frac{1}{2^{2n}} \binom{4n+1}{2n} \quad (6.25)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k} \frac{2^{2k}}{(2k+1) \binom{2k}{k}} = \frac{2^{2n}}{(2n+1) \binom{2n}{n}} \binom{x+n+\frac{1}{2}}{n} \quad (6.26)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+k}{k} \frac{2^{2k}}{(2k+1) \binom{2k}{k}} = \frac{2^{2n}}{(2n+1) \binom{2n}{n}} \binom{n-x-\frac{1}{2}}{n} \quad (6.27)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{2^{2k}}{(2k+1) \binom{2k}{k}} = \frac{(-1)^n}{2n+1} \quad (6.28)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{x}{2n-k} = \sum_{k=0}^n \binom{x}{n-k} \binom{x}{n+k} = \frac{1}{2} \left(\binom{2x}{2n} + \binom{x}{n}^2 \right) \quad (6.29)$$

$$\begin{aligned} \sum_{k=0}^{n-1} \binom{x}{k} \binom{x}{2n-1-k} &= \sum_{k=0}^{n-1} \binom{x}{n-1-k} \binom{x}{n+k} \\ &= \frac{1}{2} \binom{2x}{2n-1}, \quad n \geq 1 \end{aligned} \quad (6.30)$$

Remark 6.6 The following three identities are found in H. F. Sandham's Problem 4519 [1953, 47] of *The American Math. Monthly*, April 1954, Vol. 4, pp. 265-266.

$$\sum_{k=0}^{n-1} \frac{\binom{n-1}{k} \binom{x-1}{k}}{\binom{-n-1}{k} \binom{-x-1}{k}} = \frac{(-1)^n}{2} \frac{\binom{-x-\frac{1}{2}}{n-1}}{\binom{-x-1}{n-1} \binom{-\frac{1}{2}}{n}}, \quad n \geq 1 \quad (6.31)$$

$$\sum_{k=0}^{n-1} (-1)^k \frac{\binom{-x-1-k}{n-1-k} \binom{x-1}{k}}{\binom{n+k}{k}} = \frac{(-1)^n}{2} \frac{\binom{-x-\frac{1}{2}}{n-1}}{\binom{-\frac{1}{2}}{n}}, \quad n \geq 1 \quad (6.32)$$

$$\sum_{k=0}^{n-1} \binom{x+n-1}{n-1-k} \binom{x+n-1}{n+k} = \frac{-\binom{-x-\frac{1}{2}}{n-1} \binom{x-1+n}{n}}{2 \binom{-\frac{1}{2}}{n}}, \quad n \geq 1 \quad (6.33)$$

$$\sum_{k=0}^{r-1} \binom{z}{k} y^{r-k-1} = \sum_{j=1}^r \binom{z-j}{r-j} (y+1)^{j-1}, \quad r \geq 1 \quad (6.34)$$

$$\sum_{k=0}^{r-1} \binom{z}{k} \frac{x^{r-k}}{r-k} = \sum_{j=1}^r \binom{z-j}{r-j} \frac{(x+1)^j - 1}{j}, \quad r \geq 1 \quad (6.35)$$

$$\sum_{k=0}^n \binom{n}{k} \frac{x^{r+k}}{r+k} = \sum_{j=1}^r (-1)^{r-j} \binom{r-1}{r-j} \frac{(x+1)^{j+n} - 1}{j+n}, \quad r \geq 1 \quad (6.36)$$

$$\sum_{k=0}^n \binom{x}{k} f(k) = \sum_{j=0}^n (-1)^j \binom{n-x}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} f(k+j) \quad (6.37)$$

$$\sum_{k=0}^n (-1)^k \binom{n-x}{k} y^k (1+y)^{n-k} = \sum_{k=0}^n \binom{x}{k} y^k \quad (6.38)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{k} = \sum_{k=0}^n (-1)^k \binom{n-x}{k} \binom{y+n-k}{n} \quad (6.39)$$

Remark 6.7 The following identity, due to H. L. Krall, is from Part 2 of Problem 783, *Math. Magazine*, Vol. 44, 1971, p.41.

$$\sum_{k=0}^n \frac{1}{\binom{n}{k}} \binom{x}{k} \binom{y}{k} \binom{z}{n-k} \binom{x+y+z-k}{n-k} = \binom{x+z}{n} \binom{y+z}{n} \quad (6.40)$$

$$\sum_{k=0}^n \binom{y}{k} \binom{x+y+z-k}{n-k} \binom{x+z-n}{x-k} = \binom{x+z}{x} \binom{y+z}{n} \quad (6.41)$$

$$\sum_{k=0}^n \frac{\binom{x}{k} \binom{y}{k} \binom{z}{n-k}}{\binom{x+y+z}{k}} = \frac{\binom{x+z}{n} \binom{y+z}{n}}{\binom{x+y+z}{n}} \quad (6.42)$$

$$\sum_{k=0}^n \frac{\binom{x}{k} \binom{y}{k} \binom{x+y+z-k}{n-k}}{\binom{z-n+k}{k}} = \frac{\binom{x+z}{n} \binom{y+z}{n}}{\binom{z}{n}} \quad (6.43)$$

6.3 Equation (6.9) in Reciprocal Binomial Identities

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\binom{z}{j}}{\binom{y}{j}} = \frac{\binom{y-z}{n}}{\binom{y}{n}} \quad (6.44)$$

6.3.1 Specific Evaluations of Equation (6.44)

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{z+j}{j} \frac{1}{\binom{x+j}{j}} = \frac{\binom{-x-1-z}{n}}{\binom{-x-1}{n}} \quad (6.45)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{j+1}{\binom{x+j}{j}} = \frac{\binom{-x+1}{n}}{\binom{-x-1}{n}} \quad (6.46)$$

$$\sum_{j=0}^n \binom{n}{j} \binom{z}{j} \frac{1}{\binom{x+j}{j}} = \frac{\binom{x+z+n}{n}}{\binom{x+n}{n}} \quad (6.47)$$

$$\sum_{j=0}^n \binom{n}{j} \binom{z}{j} \frac{1}{j+1} = \frac{1}{n+1} \binom{z+n+1}{n} \quad (6.48)$$

$$\sum_{j=0}^{2n} \binom{2n}{j} \binom{2j}{j} \frac{1}{2^{2j} \binom{n+j}{j}} = \frac{1}{2^{4n}} \frac{\binom{6n}{3n}}{\binom{2n}{n}} \quad (6.49)$$

Remark 6.8 The following identity is due to Harry Bateman. The reader is referred to *The Harry Bateman Manuscript Project by Erdelyi, Magnus, and Oberhettinger. In particular, see Higher Transcendental Functions, Vol. I, [Section 2.5.3, P. 86], McGraw-Hill, 1953.*

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\binom{z}{j}}{\binom{-z+n-1}{j}} = \frac{\binom{2z}{n}}{\binom{z}{n}} \quad (6.50)$$

6.3.2 Applications of Equation (6.44)

Remark 6.9 In the following two identities, we will assume x and z are not values which make the denominator equal to zero.

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{\binom{x+k}{k}} \sum_{j=1}^k \frac{1}{j+x} = \frac{n}{(x+n)^2}, \quad n \geq 1 \quad (6.51)$$

$$\sum_{k=1}^n \binom{n}{k} \binom{z}{k} \frac{1}{\binom{x+k}{k}} \sum_{j=1}^k \frac{1}{j+x} = \frac{\binom{x+z+n}{n}}{\binom{x+n}{n}} \left(\sum_{j=1}^n \frac{1}{j+x} - \sum_{j=1}^n \frac{1}{j+x+z} \right), \quad n \geq 1 \quad (6.52)$$

$$\sum_{k=1}^n \binom{n}{k}^2 \sum_{j=1}^k \frac{1}{j} = \binom{2n}{n} \left(\sum_{j=1}^n \frac{1}{j} - \sum_{j=n+1}^{2n} \frac{1}{j} \right), \quad n \geq 1 \quad (6.53)$$

$$\sum_{k=1}^n (-1)^k \binom{n}{k} \binom{n+k-1}{k} \sum_{j=1}^k \frac{1}{j} = \frac{(-1)^n}{n}, \quad n \geq 1 \quad (6.54)$$

Remark 6.10 The following identity, due to R. R. Goldberg, is found in Problem 4805 of *The American Math. Monthly*, Vol. 65, 1958, P. 633.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k-1}{k} \sum_{j=1}^{n+k-1} \frac{1}{j} = \frac{(-1)^n}{n}, \quad n \geq 1 \quad (6.55)$$

6.4 Application of Equation (6.9) to $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x}{k} \binom{x}{2n-k}$

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x}{k} \binom{x}{2n-k} &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x+k}{k} \binom{x+2n-k}{2n-k} \\ &= (-1)^n \binom{2n}{n} \binom{x+n}{2n} = \binom{x}{n} \binom{-x-1}{n} \end{aligned} \quad (6.56)$$

6.4.1 Specific Evaluations of Equation (6.56)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{n+k}{n} \binom{3n-k}{n} = (-1)^n \binom{2n}{n} \quad (6.57)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{3n+k}{k} \binom{5n-k}{2n-k} = (-1)^n \binom{2n}{n} \binom{4n}{2n} \quad (6.58)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{3n}{k} \binom{3n}{2n-k} = (-1)^n \binom{2n}{n} \binom{4n}{2n} \quad (6.59)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \binom{4n-2k}{2n-k} = \binom{2n}{n}^2 \quad (6.60)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{1 + (-1)^n}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor}^2 \quad (6.61)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+\frac{1}{2}}{k} \binom{2n+\frac{1}{2}}{2n-k} = (-1)^n \binom{2n}{n} \binom{3n+\frac{1}{2}}{2n} \quad (6.62)$$

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \binom{4n+1}{2k} \binom{4n+1}{4n-2k} \binom{2k}{k} \binom{4n-2k}{2n-k} \frac{1}{\binom{2n}{k}} \\ = (-1)^n \frac{n+1}{2n+1} \binom{6n+1}{2n} \binom{4n+1}{3n} \end{aligned} \quad (6.63)$$

6.4.2 Applications of Equation (6.56)

Remark 6.11 *The following identity is found in L. Carlitz's "Note on a formula of Szily", Scripta Mathematica, Vol. 18 (1952), pp. 249-253.*

$$\sum_{k=0}^r (-1)^k \binom{2\alpha}{k} \binom{2r-2\alpha}{r-k}^2 = (-1)^\alpha \frac{\binom{2\alpha}{\alpha} \binom{2r-\alpha}{r}^2}{\binom{2r-\alpha}{\alpha}} \quad (6.64)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} \binom{x}{n-k} = \frac{(-1)^n + 1}{2} (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{x + \lfloor \frac{n}{2} \rfloor}{n} \quad (6.65)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} \binom{x}{n-k} = \frac{(-1)^n + 1}{2} \binom{x}{\frac{n}{2}} \binom{-x-1}{\frac{n}{2}} \quad (6.66)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+k}{k} \binom{x+n-k}{n-k} = \frac{(-1)^n + 1}{2} (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{x + \lfloor \frac{n}{2} \rfloor}{n} \quad (6.67)$$

Remark 6.12 *In the following identity, we assume x is a real or complex which is not a negative integer. We evaluate $x! = \Gamma(x+1)$.*

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 \binom{2x-n}{x-k} = \frac{(-1)^n + 1}{2} (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{x + \lfloor \frac{n}{2} \rfloor}{n} \frac{\binom{2x}{x}}{\binom{2x}{n}} \quad (6.68)$$

6.5 The Index Shift Formula and Equation (6.9)

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k+\alpha} = \binom{n+x}{n+\alpha} \quad (6.69)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k-\alpha} = \binom{n+x}{n-\alpha} \quad (6.70)$$

6.5.1 Applications of Equation (6.69)

Remark 6.13 For the following two identities, a reference is P. Tardy's, "Sopra alcune formole relativa ai coefficienti binomiali", *Battaglini's Journal*, Vol. 3 (1865), pp. 1-3. In the second identity, we assume a is a nonnegative integer.

$$\sum_{k=0}^n \binom{x+1}{k} \binom{x+1}{k+1} \binom{x-k}{n-k} = \binom{x+1}{n+1} \binom{x+n+2}{n} \quad (6.71)$$

$$\sum_{k=0}^n \binom{x+1}{k} \binom{x+a-1}{k+a-1} \binom{x-k}{n-k} = \binom{x+a-1}{n+a-1} \binom{x+n+a}{n} \quad (6.72)$$

$$\sum_{j=0}^n \binom{n}{j} \binom{x}{j} z^j = \sum_{\alpha=0}^n \binom{n}{\alpha} \binom{x+n-\alpha}{n} (z-1)^\alpha \quad (6.73)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{x}{j} = \sum_{\alpha=0}^n (-1)^\alpha \binom{n}{\alpha} \binom{x+n-\alpha}{n} 2^\alpha \quad (6.74)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{2j}{j} \left(\frac{z}{4}\right)^j = \frac{1}{2^{2n}} \sum_{\alpha=0}^n (-1)^\alpha \binom{2n-2\alpha}{n-\alpha} \binom{2\alpha}{\alpha} (z-1)^\alpha \quad (6.75)$$

$$2^{2n} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{2j}{j} = \sum_{\alpha=0}^n (-1)^\alpha \binom{2n-2\alpha}{n-\alpha} \binom{2\alpha}{\alpha} 3^\alpha \quad (6.76)$$

Remark 6.14 The following identity is from the *Mathematical Reviews*, Vol. 17, No 5., May 1956, pp. 459-460.

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{x+k}{2n} = \binom{x}{n}^2 \quad (6.77)$$

Remark 6.15 The following identity is from the *Mathematical Reviews*, Vol. 18, No 1., January 1957, P. 4.

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{x+2n-k}{2n} = \binom{x+n}{n}^2 \quad (6.78)$$

$$\sum_{k=0}^n \binom{x}{k}^2 \binom{n+2x-k}{n-k} = \binom{x+n}{n}^2 \quad (6.79)$$

$$\sum_{k=0}^n \binom{x-n}{k}^2 \binom{2x-n-k}{n-k} = \binom{x}{n}^2 \quad (6.80)$$

Remark 6.16 For the following identity see “*Remark on a Note of P. Turán*” by T. S. Nanjundiah in *American Math Monthly*, Vol. 65, No. 5, May 1958, P. 354.

$$\sum_{k=0}^n \binom{m-x+y}{k} \binom{n+x-y}{n-k} \binom{x+k}{m+n} = \binom{x}{m} \binom{y}{n}, \quad m \text{ integral, } m \geq 0 \quad (6.81)$$

Remark 6.17 The following identity is a special case of Saalschütz’s Theorem. See the review by L. Carlitz in *Math Reviews*, Vol. 18, No. 1. January 1957, P. 4.

$$\sum_{k=0}^n \binom{n}{k} \binom{\alpha}{k} \binom{x+n+\alpha-k}{n+\alpha} = \binom{x+\alpha}{\alpha} \binom{x+n}{n} \quad (6.82)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{-x+r-1}{r-j} \binom{x-r-n}{\alpha-r-j} = (-1)^r \binom{\alpha+n}{\alpha-r} \binom{x}{\alpha} \quad (6.83)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{-x-1}{j} \binom{x-n}{n-j} = \binom{2n}{n} \binom{x}{n} \quad (6.84)$$

$$\sum_{j=0}^n \binom{n}{j}^2 \binom{2n+j}{2n} = \sum_{j=0}^n \binom{n}{j}^2 \binom{3n-j}{2n} = \binom{2n}{n}^2 \quad (6.85)$$

$$\sum_{j=0}^n \binom{n}{j} \binom{3n+j}{j} \binom{2n}{n-j} = \binom{2n}{n} \binom{3n}{n} \quad (6.86)$$

$$\sum_{j=0}^n \binom{n}{j}^2 \binom{3n+j}{2n} = \binom{3n}{n}^2 \quad (6.87)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j}^2 \binom{2j}{j} \binom{4n-2j}{2n-j} \frac{1}{\binom{2n}{j}} = \binom{2n}{n}^2 \quad (6.88)$$

$$\sum_{j=0}^n \binom{n}{j}^2 \binom{4n+2j+1}{2j} \frac{1}{(2j+1)\binom{2n+j}{j}} = \frac{1}{2n+1} \binom{4n+1}{2n} \quad (6.89)$$

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} \binom{-x-1}{j} \binom{x-2n}{\alpha-j} = \binom{\alpha+2n}{\alpha} \binom{x}{\alpha} \quad (6.90)$$

$$\sum_{j=0}^n \binom{2n}{j} \binom{3n+j}{3n} \binom{n}{j} = \binom{3n}{n}^2 \quad (6.91)$$

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{x}{k} \binom{y}{k+r}}{\binom{x+y+n}{k}} = \frac{\binom{x+r+n}{n} \binom{y+n}{n+r}}{\binom{x+y+n}{n}} \quad (6.92)$$

Remark 6.18 The following identity solves a problem on Page 122, Vol. 29, 1947, of *Norsk Matematisk Tidsskrift*.

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{x}{k} \binom{y}{k}}{\binom{x+y+n}{k}} = \frac{\binom{x+n}{n} \binom{y+n}{n}}{\binom{x+y+n}{n}} \quad (6.93)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k} \binom{x+n+k+\alpha}{k+\alpha} = \binom{x+\alpha+n}{n} \binom{x+\alpha+n}{n+\alpha} \quad (6.94)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{r}{k} \binom{x+n+r+k}{n+r} = \binom{x+n+r}{n} \binom{x+n+r}{r} \quad (6.95)$$

6.6 The Minus One Transformation Applied to Equation (6.9)

$$\sum_{k=0}^n \binom{x+k}{k} \binom{y+n-k}{n-k} = \binom{x+y+n+1}{n} \quad (6.96)$$

$$\sum_{k=\alpha}^{n-\beta} \binom{k}{\alpha} \binom{n-k}{\beta} = \binom{n+1}{\alpha+\beta+1}, \quad n-\beta \geq \alpha \quad (6.97)$$

7 Alternating Convolutions

Remark 7.1 Throughout this chapter, we assume, unless otherwise specified, that n and r are nonnegative integers, while x , y , and z are arbitrary real or complex numbers.

7.1 Evaluation of $\sum_{k=0}^{2n} (-1)^k \binom{x}{k} \binom{x}{2n-k}$

$$\sum_{k=0}^{2n} (-1)^k \binom{x}{k} \binom{x}{2n-k} = (-1)^n \binom{x}{n} \quad (7.1)$$

Remark 7.2 In the following identity, we evaluate $x! = \Gamma(x+1)$.

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2x}{k+x} \binom{k+x}{2n} = (-1)^n \binom{x}{n} \binom{2x}{x} \quad (7.2)$$

7.1.1 Specific Evaluations and Applications of Equation (7.1)

$$\sum_{k=0}^{2n} (-1)^k \binom{2k}{k} \binom{4n-2k}{2n-k} = 2^{2n} \binom{2n}{n} \quad (7.3)$$

$$\sum_{k=0}^n (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{1+(-1)^n}{2} \binom{n}{\frac{n}{2}} 2^n \quad (7.4)$$

$$\sum_{k=0}^n \binom{4k}{2k} \binom{4n-4k}{2n-2k} = 2^{4n-1} + 2^{2n-1} \binom{2n}{n} \quad (7.5)$$

$$\sum_{k=0}^{n-1} \binom{4k+2}{2k+1} \binom{4n-2k-2}{2n-2k-1} = 2^{4n-1} - 2^{2n-1} \binom{2n}{n}, \quad n \geq 1 \quad (7.6)$$

$$\sum_{k=0}^n (-1)^k \binom{x}{n-k} \binom{x}{n+k} = \frac{1}{2} \left(\binom{x}{n} + \binom{x}{n}^2 \right) \quad (7.7)$$

$$\sum_{k=0}^n (-1)^k \frac{\binom{n}{k} \binom{x}{k}}{\binom{n+k}{k} \binom{x+k}{k}} = \frac{1}{2} \left(1 + \frac{1}{\binom{x+n}{n}} \right) \quad (7.8)$$

7.2 Evaluation of $\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{x}{n-k}$

Remark 7.3 Recall that for real x , $[x]$ denotes the integer part of x .

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{x}{n-k} = \frac{(-1)^n + 1}{2} (-1)^{[\frac{n}{2}]} \binom{x}{[\frac{n}{2}]} \quad (7.9)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2x-n}{x-k} = \frac{(-1)^n + 1}{2} (-1)^{[\frac{n}{2}]} \binom{x}{[\frac{n}{2}]} \frac{\binom{2x}{x}}{\binom{2x}{n}}, \quad \text{see Remark 7.2} \quad (7.10)$$

$$\sum_{k=0}^n (-1)^k \binom{x+k}{k} \binom{x+n-k}{n-k} = (-1)^n \binom{x+\frac{n}{2}}{\frac{n}{2}} \frac{1 + (-1)^n}{2} \quad (7.11)$$

7.2.1 Applications of Equation (7.9)

$$\sum_{k=0}^n (-1)^k \binom{y}{n-k} \binom{x+y}{k} = \sum_{\alpha=0}^n (-1)^{\alpha+[\frac{n-\alpha}{2}]} \binom{x}{\alpha} \frac{(-1)^{n-\alpha} + 1}{2} \binom{y}{[\frac{n-\alpha}{2}]} \quad (7.12)$$

$$\sum_{k=r}^{n-r} (-1)^k \binom{k}{r} \binom{n-k}{r} = (-1)^r \frac{1 + (-1)^n}{2} \binom{\frac{n}{2}}{r}, \quad n \geq 2r \quad (7.13)$$

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{y}{n-k} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x}{k} \binom{y-x}{n-2k} \quad (7.14)$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x}{k} \binom{-x}{n-2k} = (-1)^n \binom{x}{n} \quad (7.15)$$

Remark 7.4 In the following two identities, we evaluate $z!$ by $\Gamma(z+1)$, whenever z is a real (complex) number which is not a negative integer. A reference for these two identities is H. Bateman's, "Higher Transcendental Functions", Vol I, Chapter II [editor W. Magnus].

$$\sum_{k=0}^n \binom{n}{k} \binom{z}{k} \frac{1}{(-z+n-1)_k} = \frac{2^n (z-n)! \sqrt{\pi}}{(z-\frac{n}{2})! (\frac{-n}{2}-\frac{1}{2})!} \quad (7.16)$$

$$\sum_{k=0}^n (-1)^k \binom{z}{k} \binom{z}{n-k} = \binom{z}{n} \frac{2^n (z-n)! \sqrt{\pi}}{(z-\frac{n}{2})! (\frac{-n}{2}-\frac{1}{2})!} \quad (7.17)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{x}{2k} \binom{x}{n-2k} = \frac{1}{2} \binom{2x}{n} + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} \binom{x}{\lfloor \frac{n}{2} \rfloor} (-1)^n + 1}{2} \quad (7.18)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{x}{2k+1} \binom{x}{n-2k-1} = \frac{1}{2} \binom{2x}{n} - \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} \binom{x}{\lfloor \frac{n}{2} \rfloor} (-1)^n + 1}{2}, \quad n \geq 1 \quad (7.19)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}^2 = \frac{1}{2} \binom{2n}{n} + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} (-1)^n + 1}{2} \quad (7.20)$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1}^2 = \frac{1}{2} \binom{2n}{n} - \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} (-1)^n + 1}{2}, \quad n \geq 1 \quad (7.21)$$

7.3 Quotient Identities Involving Equations (6.9) and (7.9)

Remark 7.5 Throughout this section, the reader may let x be a real or complex number. Then $x!$ is evaluated as $\Gamma(x+1)$.

7.3.1 First Example of a Reciprocal Identity

$$\sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+2x}{k+x}} f(k) = \frac{(-1)^n}{\binom{2x}{x} \binom{n+2x}{n}} \sum_{k=0}^n \binom{-x-1}{k} \binom{-x-1}{n-k} f(k) \quad (7.22)$$

Specific Evaluations of Equation (7.22)

$$\sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+2x}{k+x}} = \frac{2x+n+1}{(2x+1)\binom{2x}{x}} \quad (7.23)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{n+2x}{k+x}} = \frac{(-1)^{n+\lceil \frac{n}{2} \rceil}}{\binom{2x}{x} \binom{2x+n}{n}} \frac{1+(-1)^n}{2} \binom{-x-1}{\frac{n}{2}} \quad (7.24)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{1}{\binom{2x+2n}{x+k}} = \frac{\binom{x+n}{n}}{\binom{2x}{x} \binom{2x+2n}{2n}} = \frac{\binom{2n}{n}}{\binom{x+n}{n} \binom{2x+2n}{x+n}} \quad (7.25)$$

7.3.2 Second Example of a Reciprocal Identity

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2x}{k+x} = (-1)^n \binom{2n}{n} \frac{\binom{2n+2x}{n+x}}{\binom{2n+x}{n}} = (-1)^n \binom{2n}{n} \frac{\binom{2n+2x}{n}}{\binom{n+x}{n}} \quad (7.26)$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \binom{2n+2x}{k+x} = \binom{4n+2x}{2n+x} \quad (7.27)$$

7.3.3 Third Example of a Reciprocal Identity

$$\sum_{k=0}^n \binom{n}{k} \binom{n+2x}{k+x} f(k) = \frac{\binom{2n+2x}{n+x}}{\binom{2n+2x}{n}} \sum_{k=0}^n \binom{n+x}{k} \binom{n+x}{n-k} f(k) \quad (7.28)$$

Specific Evaluations of Equation (7.28)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+2x}{k+x} = \frac{\binom{2n+2x}{n+x}}{\binom{2n+2x}{n}} (-1)^{\lceil \frac{n}{2} \rceil} \frac{1+(-1)^n}{2} \binom{-n-x}{\frac{n}{2}} \quad (7.29)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{n+2x}{k+x} = \binom{2n+2x}{n+x} \quad (7.30)$$

7.4 Evaluation of $\sum_{k=0}^{2n} (-1)^k \binom{y}{2n-k} \binom{z}{k}$

$$\sum_{k=0}^{2n} (-1)^k \binom{y}{2n-k} \binom{z}{k} = \sum_{\alpha=0}^n (-1)^\alpha \binom{y}{\alpha} \binom{z-y}{2n-2\alpha} \quad (7.31)$$

$$\sum_{k=0}^{2n} \binom{\frac{-1}{2}}{k} = \frac{1}{2^{2n}} \sum_{\alpha=0}^n \binom{2\alpha}{\alpha} \binom{4n-4\alpha}{2n-2\alpha} 2^{2\alpha} \quad (7.32)$$

8 An Introduction to the n^{th} Difference Operator

Remark 8.1 Throughout this chapter, we will assume, unless otherwise specified, that n and r are nonnegative integers, x is a real or complex number, and h is nonzero real or complex number.

8.1 Definition and Basic Properties of $\Delta_h^n f(x)$

8.1.1 Definition of $\Delta_h^n f(x)$

$$\Delta_h^n f(x) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \frac{f(x+kh)}{h^n} \quad (8.1)$$

Relationship to Derivative

$$\frac{d^n f(x)}{dx^n} = \lim_{h \rightarrow 0} \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \frac{f(x+kh)}{h^n} \quad (8.2)$$

Linearity Properties of Δ_h^n

$$\Delta_h^1(f(x) + \varphi(x)) = \Delta_h^1 f(x) + \Delta_h^1 \varphi(x) \quad (8.3)$$

$$\Delta_h^n \Delta_h^r f(x) = \Delta_h^r \Delta_h^n f(x) = \Delta_h^{n+r} f(x) \quad (8.4)$$

$$\Delta_h^1(cf(x)) = c\Delta_h^1 f(x), \quad c \text{ is a constant} \quad (8.5)$$

Two Basic Examples

Remark 8.2 In the following two examples, we assume a and α are nonzero real or complex numbers.

$$\Delta_h^n a^x = a^x \left(\frac{a^h - 1}{h} \right)^n \quad (8.6)$$

$$\Delta_h^n (1 + \alpha h)^{\frac{x}{h}} = (1 + \alpha h)^{\frac{x}{h}} \alpha^n \quad (8.7)$$

8.1.2 Inversion of Equation (8.1)

$$f(x + nh) = \sum_{k=0}^n \binom{n}{k} (-1)^k \Delta_h^k f(x) \quad (8.8)$$

8.1.3 Calculation of $\Delta_1^k \binom{x}{k} \frac{k!^p}{x^{kp}}$

Remark 8.3 Throughout this section, we assume p is an integer while x is a nonzero real or complex number.

$$\sum_{k=0}^n \binom{x}{k}^p \frac{k!^p}{x^{(k+1)p}} ((x-k)^p - x^p) = \binom{x}{n+1}^p \frac{(n+1)!^p}{x^{(n+1)p}} - 1 \quad (8.9)$$

$$\sum_{k=0}^n \binom{x}{k} \frac{k k!}{x^{k+1}} = 1 - \binom{x}{n+1} \frac{(n+1)!}{x^{n+1}} \quad (8.10)$$

Remark 8.4 The following two identities are found in D. Steinberg's "Combinatorial Derivations of Two Identities", *Mathematics Magazine*, Vol. 31, No.4, 1958, pp. 207-9.

$$\sum_{k=1}^m \frac{m! k m^{n-k-1}}{(m-k)!} = m^n, \quad \text{for } m < n+1 \quad (8.11)$$

$$\sum_{k=1}^{n-1} \frac{m! k m^{n-k-1}}{(m-k)!} + \frac{m!}{(m-n)!} = m^n, \quad \text{for } m > n \quad (8.12)$$

$$\sum_{k=0}^n k k! = (n+1)! - 1 \quad (8.13)$$

$$\sum_{k=0}^n \binom{2k}{k} \frac{k k!}{2^k} = \binom{2n+2}{n+1} \frac{(n+1)!}{2^{n+2}} - \frac{1}{2} \quad (8.14)$$

$$\sum_{k=0}^n (-1)^k \binom{-x+k-1}{k} \frac{k k!}{x^k} = x + (-1)^n \binom{-x+n}{n+1} \frac{(n+1)!}{x^n} \quad (8.15)$$

$$\sum_{k=0}^n \binom{n+k}{k} \frac{k k!}{(n+1)^k} = \binom{2n+1}{n+1} \frac{(n+1)!}{(n+1)^n} - n - 1 \quad (8.16)$$

$$\sum_{k=0}^n \frac{x^{k+1} k}{x(x-k) k! \binom{x}{k}} = \frac{x^{n+1}}{(n+1)! \binom{x}{n+1}} - 1 \quad (8.17)$$

$$\sum_{k=0}^n \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!} \quad (8.18)$$

8.1.4 Euler's Transformation

If

$$(-1)^n \Delta_1^n f(0) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(k), \quad (8.19)$$

then

$$\sum_{n=0}^{\infty} (-1)^n f(n) = \sum_{n=0}^{\infty} (-1)^n \frac{\Delta_1^n f(0)}{2^{n+1}}. \quad (8.20)$$

8.1.5 Alternative Definition of the n^{th} Difference Operator

Definition of $\tilde{\Delta}_h^n$

$$\tilde{\Delta}_h^n = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x + kh) \quad (8.21)$$

Inversion of Equation (8.21)

$$f(x + nh) = \sum_{k=0}^n (-1)^k \binom{n}{k} \tilde{\Delta}_h^n f(x) \quad (8.22)$$

$$\sum_{j=0}^n \tilde{\Delta}_h^n f(x) = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} f(x + kh) \quad (8.23)$$

$$\sum_{j=0}^n f(x + jh) = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \tilde{\Delta}_h^k f(x) \quad (8.24)$$

Examples of Equations (8.21) and (8.22)

$$\sum_{j=1}^n \frac{1}{j} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k}, \quad n \geq 1 \quad (8.25)$$

$$\frac{1}{n} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{j=1}^k \frac{1}{j}, \quad n \geq 1 \quad (8.26)$$

Example of Equation (8.23) Let

$$f(n) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} F(k). \quad (8.27)$$

Then,

$$\sum_{i=1}^n f(i) = \sum_{k=1}^n (-1)^{k-1} \binom{n+1}{k+1} F(k). \quad (8.28)$$

Generalization of Equations (8.21) and (8.22): The Inversion Pair Theorem

$$\sum_{k=\alpha}^n (-1)^k \binom{k}{\alpha} f(k) = (-1)^n g(\alpha) \quad (8.29)$$

if and only if

$$\sum_{k=\alpha}^n (-1)^k \binom{k}{\alpha} g(k) = (-1)^n f(\alpha) \quad (8.30)$$

8.2 The Shift Operator E_h^n

8.2.1 Definition of E_h^n

$$E_h^n f(x) = f(x + nh) \quad (8.31)$$

8.2.2 Relationships Between the Shift and Difference Operators

$$\Delta_h^n f(x) = \frac{(-1)^n}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} E_h^k f(x) \quad (8.32)$$

$$\Delta_h^n f(x) = \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} E_h^{n-k} f(x) \quad (8.33)$$

$$E_h^n f(x) = \sum_{k=0}^n \binom{n}{k} h^k \Delta_h^k f(x) \quad (8.34)$$

$$E_h^1 \Delta_h^1 f(x) = \Delta_h^1 E_h^1 f(x) = \frac{f(x + 2h) - f(x + h)}{h} \quad (8.35)$$

8.2.3 Product Formula for $\Delta_h^n uv$

$$\Delta_h^n uv = \sum_{k=0}^n \binom{n}{k} \Delta_h^k u \Delta_h^{n-k} (E_h^k v) \quad (8.36)$$

Application of Equation (8.36)

Remark 8.5 *In the following example, we assume a is a nonzero real or complex number.*

$$\Delta_h^n (e^{ax} f(x)) = e^{ax} \sum_{k=0}^n \binom{n}{k} e^{ahk} \Delta_h^k f(x) \left(\frac{e^{ah} - 1}{h} \right)^{n-k} \quad (8.37)$$

8.2.4 Recurrence Formula for $\Delta_h^n f(x + h)$

$$\Delta_h^n f(x + h) = h \Delta_h^{n+1} f(x) + \Delta_h^n f(x) \quad (8.38)$$

9 Calculations Involving $(-1)^n \Delta_1^n \frac{1}{x} \Big|_{x=r}$

Remark 9.1 Throughout this chapter, we assume n and m are nonnegative integers. We also assume, unless otherwise specified r, z, x , and a are real or complex numbers for which the Gamma function is defined.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+r} = \frac{n!(r-1)!}{(n+r)!} = \beta(r, n+1) \quad (9.1)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{ak+1} = \frac{1}{\binom{n+\frac{1}{a}}{n}} \quad (9.2)$$

9.1 Specific Evaluations of Equation (9.2)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{2k+1} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \quad (9.3)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{2n-2k+1} = (-1)^n \frac{2^{2n}n!^2}{(2n+1)!} \quad (9.4)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{2k}k!^2}{(2k+1)!} = \frac{1}{2n+1} \quad (9.5)$$

9.2 Applications of the Inversion Pair to Equation (9.2)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{k+\frac{1}{a}}{k}} = \frac{1}{an+1}, \quad a \neq 0 \quad (9.6)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{k+m}{k}} = \frac{m}{n+m} \quad (9.7)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{k+\frac{1}{3}}{k}} = \frac{1}{3n+1} \quad (9.8)$$

$$\sum_{k=0}^n (-1)^k \binom{n+r}{n-k} = \binom{n+r-1}{n} \quad (9.9)$$

$$\sum_{j=0}^n \frac{1}{aj+1} = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \frac{1}{\binom{k+\frac{1}{a}}{k}}, \quad a \neq 0 \quad (9.10)$$

$$\sum_{j=1}^n \frac{1}{j+a} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k \binom{k+a}{a}} \quad (9.11)$$

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{j=1}^k \frac{1}{j+a} = \frac{1}{n \binom{n+a}{a}} \quad (9.12)$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{j=1}^k \frac{1}{j+a} = \frac{1}{a}, \quad a \neq 0 \quad (9.13)$$

9.2.1 Applications of Equation (9.7)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{j=1}^m \frac{1}{k+j} = \frac{1}{n} \left(1 - \frac{1}{\binom{n+m}{n}} \right), \quad n \geq 1, m \geq 1 \quad (9.14)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{z+k} = \frac{e^{-1}}{z} \sum_{j=0}^{\infty} \frac{1}{\binom{z+j}{j} j!}, \quad z \neq 0 \quad (9.15)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{n+1+k} = n! \left(1 - \frac{1}{e} \sum_{k=0}^n \frac{1}{k!} \right) \quad (9.16)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{n+k}{k} k!} = n! \left(e - \sum_{k=0}^n \frac{1}{k!} \right) \quad (9.17)$$

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{\binom{z+k}{k}} = \sum_{j=0}^{\infty} (-1)^j \frac{z}{z+j} \left(\frac{x}{1-x} \right)^{j+1}, \quad |x| < 1, \quad \left| \frac{x}{1-x} \right| < 1 \quad (9.18)$$

Two Restatements of Equation (9.18)

$$\sum_{j=0}^{\infty} \frac{x^{j+1}}{z+j} = -\sum_{k=0}^{\infty} \left(\frac{x}{x-1}\right)^{k+1} \frac{1}{z\binom{z+k}{k}}, \quad \left|\frac{x}{1-x}\right| < 1 \quad (9.19)$$

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{\binom{z}{k}} = \sum_{j=0}^{\infty} \left(\frac{x}{x+1}\right)^{j+1} \frac{z+1}{z+1-j}, \quad \left|\frac{x}{1+x}\right| < 1 \quad (9.20)$$

Three Specific Evaluations of Equation (9.18)

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{\binom{z+k}{k}} = \sum_{j=0}^{\infty} \frac{z}{2^{j+1}(z+j)} \quad (9.21)$$

$$\sum_{k=0}^{\infty} \frac{1}{\binom{x+k}{k} 2^k} = 2 \sum_{j=0}^{\infty} (-1)^j \frac{x}{x+j} \quad (9.22)$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{x+j} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{1}{\binom{x+k}{k} 2^{k+1}}, \quad x \neq 0 \quad (9.23)$$

10 Euler's Finite Difference Theorem

Remark 10.1 Throughout this chapter, we assume n, j, m, p, α , and r are nonnegative integers. We also assume x, z, a, y , and b are real or complex numbers.

10.1 Statement of Euler's Finite Difference Theorem

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k^j = \begin{cases} 0, & 0 \leq j < n \\ (-1)^n n!, & j = n \end{cases} \quad (10.1)$$

10.2 Variations of Equation (10.1)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k^{n+1} = (-1)^n (n+1)! \frac{n}{2} \quad (10.2)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k^{n+2} = (-1)^n (n+2)! \frac{n(3n+1)}{24} \quad (10.3)$$

10.3 Polynomial Extension of Equation (10.1)

Remark 10.2 Given a polynomial $\sum_{j=0}^r a_j k^j$ of degree r in k , with a_j free of k , Equation (10.1) implies

$$\sum_{j=0}^r a_j \sum_{k=0}^n (-1)^k \binom{n}{k} k^j = \begin{cases} 0, & r < n \\ (-1)^n n! a_n, & r = n \end{cases} \quad (10.4)$$

10.3.1 Applications of Equation (10.4)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (x - k)^r = \begin{cases} 0, & r < n \\ n!, & r = n \end{cases} \quad (10.5)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (a - bk)^r = \begin{cases} 0, & r < n \\ b^n n!, & r = n \end{cases} \quad (10.6)$$

Remark 10.3 Throughout the remainder of this chapter, we will assume, unless otherwise specified, that $f(x)$ is a polynomial of degree n in x , namely $f(x) = \sum_{i=0}^n a_i x^i$.

$$\sum_{k=0}^m (-1)^k \binom{m}{k} f(x - kz) = \begin{cases} 0, & n < m \\ a_m z^m m!, & n = m \end{cases} \quad (10.7)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{m}{r} = \begin{cases} 0, & r < n \\ (-1)^n m^n, & r = n, \end{cases} \quad m \text{ not necessarily an integer} \quad (10.8)$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j}{r} = \begin{cases} 0, & r < n \\ (-1)^n, & r = n \end{cases} \quad (10.9)$$

Two Applications of Equation (10.9)

$$\sum_{k=1}^n k^p = \sum_{k=1}^p (-1)^k \binom{n+1}{k+1} \sum_{j=1}^k (-1)^k \binom{k}{j} j^p, \quad p \geq 1, n \geq 1 \quad (10.10)$$

$$\sum_{k=1}^n k^p = \sum_{k=0}^p (-1)^k \binom{n}{k+1} \sum_{j=0}^k (-1)^k \binom{k}{j} (j+1)^p, \quad n \geq 1 \quad (10.11)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{r+k}{r} = \begin{cases} 0, & r < n \\ (-1)^n, & r = n \end{cases} \quad (10.12)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y+xk}{r} = \begin{cases} 0, & r < n \\ (-1)^n x^n, & r = n, \end{cases} \quad (10.13)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{2xk}{2\alpha}}{\binom{xk}{\alpha}} = \begin{cases} 0, & 0 \leq \alpha < n \\ (-1)^n x^n \frac{2^{2n}}{\binom{2n}{n}}, & \alpha = n \end{cases} \quad (10.14)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+k}{r} = (-1)^n \binom{x}{r-n} \quad (10.15)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{\alpha+k-1}{\alpha} = \begin{cases} 0, & \alpha < n \\ (-1)^n \binom{\alpha-1}{n-1}, & \alpha \geq n \end{cases} \quad (10.16)$$

$$\Delta_1^n \binom{x}{r} = \binom{x}{r-n} \quad (10.17)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+k}{r+k} = (-1)^n \binom{x}{r+n} \quad (10.18)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{-x+\alpha-1}{k+r} \binom{x}{\alpha-r-k} = (-1)^r \binom{\alpha+n}{\alpha-r} \binom{x}{\alpha} \quad (10.19)$$

Three Applications of Equation (10.15)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x-k}{j} = \binom{x-n}{j-n} \quad (10.20)$$

Remark 10.4 The following two identities are due to the work of H. Bateman and T. Ono. For references see Question 2221 in *Nouv. Ann. de Math*, Vol.4. No. 14, 1915, pp. 191-192, and Problem 2238, *Nouv. Ann. de Math*, Vol.4. No. 15, 1915, pp. 56-57. Also see “Notes on Binomial Coefficients” by H. Bateman, P. 54.

$$\sum_{k=0}^{n-a} \binom{n-a}{k} \frac{\binom{a-\frac{1}{2}+k}{a+k}}{\binom{n-\frac{1}{2}}{a+k}} = \frac{\binom{a-\frac{1}{2}}{a}}{\binom{n-\frac{1}{2}}{n}}, \quad a \text{ a nonnegative integer, } a \leq n \quad (10.21)$$

$$\sum_{k=a}^n \binom{n-a}{k-a} \frac{\binom{n}{2k}}{\binom{2n}{2k}} = 2^{2n-2a} \frac{\binom{2a}{a}}{\binom{2n}{n}}, \quad a \text{ a nonnegative integer, } a \leq n \quad (10.22)$$

$$\sum_{k=1}^n \binom{n}{k} \frac{(yk)^{n-k}(x-yk)^k}{k} = \frac{x^n}{n}, \quad n \geq 2 \quad (10.23)$$

$$\sum_{k=1}^n \binom{n}{k} (yk)^{n-k}(x-yk)^{k-1} = x^{n-1}, \quad n \geq 1 \quad (10.24)$$

$$\sum_{k=1}^n \binom{n}{k} k^{n-k}(x-k)^{k-1} = x^{n-1}, \quad n \geq 1 \quad (10.25)$$

Application of Equation (10.24)

$$\sum_{j=0}^n a_j x^j = f(x) = \sum_{k=0}^n (x+yk+y)^k \sum_{j=k}^n \binom{j+1}{k+1} (-yk-y)^{j-k} a_j \quad (10.26)$$

Abel's Sum

$$(x+y+nz)^n = x \sum_{j=0}^n \binom{n}{j} (y+(n-j)z)^{n-j} (x+jz)^{j-1} \quad (10.27)$$

Application of Equation (10.24) to an Expansion of e^{xz}

$$e^{xz} = x \sum_{k=0}^{\infty} \frac{(x+yk)^{k-1} z^k e^{-y kz}}{k!} \quad (10.28)$$

Remark 10.5 The following identity is M. S. Klamkin's solution to Problem 4489 [1952, 332] of *The American Math. Monthly*, Sept. 1953, P. 485.

$$\sum_{k=1}^{\infty} \frac{((x+yk)w)^{k-1}}{k!} = \frac{e^{xz} - 1}{wx}, \quad |w| < \frac{1}{|y|e}, \quad x \neq 0 \quad (10.29)$$

$$\sum_{k=1}^{\infty} \frac{(ywk)^{k-1}}{k!} = \frac{z}{w}, \quad w = ze^{-yz} \quad (10.30)$$

$$\sum_{k=1}^{\infty} \frac{(xe^{-x})^{k-1} k^{k-1}}{k!} = e^x \quad (10.31)$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{k}{e}\right)^{k-1}}{k!} = e \quad (10.32)$$

$$\sum_{k=1}^{\infty} \frac{k^k}{k!} (xe^{-x})^{k-1} = \frac{e^x}{1-x}, \quad x \neq 1 \quad (10.33)$$

Extension of Equation (10.32)

$$\sum_{k=0}^{\infty} \frac{x^{k-1}}{\binom{x+k}{k} k!} = \sum_{k=1}^{\infty} \frac{k^k}{k!} \frac{e^{-k}}{x+k} \quad (10.34)$$

Application of Equation (10.33)

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^{n-k} (k+1)^k = n! \sum_{j=0}^n \frac{1}{j!} \quad (10.35)$$

Three Applications of Equation (10.35)

Remark 10.6 The following identity is from Problem E1318 of *The American Math, Monthly*, Vol. 65, 1958, P. 366. Recall that for real x , $[x]$ denotes the floor of x .

$$\sum_{k=0}^n \frac{1}{k!} = \frac{[en!]}{n!}, \quad (10.36)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} [ek!] = (-1)^n n! + 1 \quad (10.37)$$

$$\sum_{k=1}^n (-1)^k \binom{n}{k} k^n \left(1 + \frac{1}{k}\right)^k = (-1)^n [en!], \quad n \geq 1 \quad (10.38)$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{n+k}{k} k!} = en! - [en!], \quad n \geq 1 \quad (10.39)$$

$$\sum_{k=1}^n k [ek!] = [e(n+1)!] - n - 2, \quad n \geq 1 \quad (10.40)$$

11 Newton-Gregory Expansions for Polynomials

Remark 11.1 Throughout this chapter, we will assume $f(x)$ is a polynomial of degree n , namely, $f(x) = \sum_{i=0}^n a_i x^i$. We will also assume, unless otherwise specified, that x and y are real or complex numbers, while n is a nonnegative integer.

Newton-Gregory Expansions

$$f(x + y) = \sum_{k=0}^n \binom{x}{k} \Delta_1^k f(z)|_{z=y} \quad (11.1)$$

$$f(x + y) = \sum_{k=0}^n \binom{\frac{x}{h}}{k} \Delta_h^k f(y), \quad \text{where } h \text{ is a nonzero real or complex number} \quad (11.2)$$

12 Recursive Formula for Functions Defined by n^{th} Differences

Remark 12.1 Throughout this chapter, we will assume n is a nonnegative integer. We also let, for real x , $[x]$ denote the floor of x .

12.1 First Recursive Formula

If

$$F(n) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{f(k)}{n-k}, \quad (12.1)$$

then

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{f(k+1)}{k+1} = \frac{f(0)}{n+2} - F(n+2). \quad (12.2)$$

Note that $f(k)$ independent of n .

12.2 Second Recursive Formula

If

$$F(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(k), \quad (12.3)$$

then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(k+1)}{k+1} = \frac{f(0) - F(n+1)}{n+1}. \quad (12.4)$$

Note that $f(k)$ independent of n .

12.3 Third Recursive Formula

Remark 12.2 Throughout this section, we assume r and α are integers.

Define

$$F_n(r+1) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(k+r+1)}{\binom{k+r+1}{k}}, \quad r \geq -1. \quad (12.5)$$

Then,

$$F_n(r+1) = \frac{r+1}{n+1} \left(f(r) - \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{f(k+r)}{\binom{k+r}{k}} \right). \quad (12.6)$$

Also,

$$\begin{aligned} F_n(\alpha+1) &= (-1)^\alpha \frac{\alpha+1}{\binom{n+\alpha}{\alpha}} \left(F_{n+\alpha}(1) - \sum_{j=0}^{\alpha-1} (-1)^j \binom{n+\alpha}{j} \frac{f(j+1)}{j+1} \right) \\ &= (-1)^\alpha \frac{\alpha+1}{\binom{n+\alpha}{\alpha}} \sum_{j=\alpha}^{n+\alpha} (-1)^j \binom{n+\alpha}{j} \frac{f(j+1)}{j+1}, \quad \alpha \geq 1. \end{aligned} \quad (12.7)$$

12.3.1 Applications of Equation (12.7)

$$F_n(n+1) = (-1)^n \frac{n+1}{\binom{2n}{n}} \sum_{j=n}^{2n} (-1)^j \binom{2n}{j} \frac{f(j+1)}{j+1} \quad (12.8)$$

Remark 12.3 The following identity, due to H. F. Sandham, is from Advanced Problem 4519 of The American Math. Monthly, January 1953, Vol. 60, No. 1, P. 47.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{(2k+1) \binom{k+n+1}{k}} = \frac{1}{4(n+1)} \left(\frac{(n+1)!^2 \cdot 2^{2n+2}}{(2n+2)!} \right)^2 \quad (12.9)$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{1}{2n-2k+1} = (-1)^n \frac{2^{4n+1}}{(2n+1) \binom{2n}{n}} \quad (12.10)$$

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{1}{2n-2k+1} = (-1)^n \frac{2^{4n}}{(2n+1) \binom{2n}{n}} \quad (12.11)$$

$$\sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \binom{k+\alpha+1}{k} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 0, & n \geq 2 \end{cases} \quad \alpha \text{ is a real or complex number} \quad (12.12)$$

Combinatorial Identities: Table II: Advanced Techniques for Summing Finite Series

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 Melzak's Formula

1.1 Statement of Melzak's Formula

Remark 1.1 Throughout this chapter, we assume n and α are nonnegative integers, and x , y , and z are real or complex numbers. Furthermore, we use the convention that $f(x)$ represents a polynomial of degree n .

Z. A. Melzak's Formula from Problem 4458 of The American Math. Monthly, 1951, P. 636

Let

$$f(x) = \sum_{i=0}^n a_i x^i. \quad (1.1)$$

Then,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k} = \frac{f(x+y)}{y \binom{y+n}{n}}, \quad y \neq 0, -1, \dots, -n. \quad (1.2)$$

1.2 Basic Examples of Melzak's Formula

1.2.1 First Example: $f(x) \equiv 1$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{z+yk} = \frac{1}{z \binom{n+\frac{z}{y}}{n}}, \quad z \neq 0, -y, -2y, \dots, -yn \quad (1.3)$$

$$\sum_{k=0}^{\alpha n+1} (-1)^k \binom{\alpha n+1}{k} \frac{1}{\alpha^2 n - \alpha n + \alpha - 1 - \alpha k} = \frac{(-1)^{\alpha n}}{(\alpha n+1) \binom{\alpha^2 n - \alpha n + \alpha - 1}{\alpha}_{\alpha n+1}}, \quad \alpha \geq 2 \quad (1.4)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{n^2 - 2n - (n-1)k} = \frac{(-1)^{n-1}}{n \binom{n^2-2n}{n-1}}, \quad n \geq 3 \quad (1.5)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{2n+1-2k}{(2n+1-nk)(2n^2-n-1-nk)} \\ = \frac{1}{n(2n+1) \binom{2n^2-n-1}{2n+1}}, \quad n \geq 2 \end{aligned} \quad (1.6)$$

Inversion of Equation (1.3)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{k+\frac{z}{y}}{k}} = \frac{z}{z+yn}, \quad z \neq -yn \quad (1.7)$$

$$\sum_{k=0}^{\alpha n+1} (-1)^k \binom{\alpha n+1}{k} \frac{1}{\binom{k+\frac{\alpha n+1}{\alpha}}{k}} = \frac{1}{1+\alpha}, \quad \alpha \geq 1 \quad (1.8)$$

$$\sum_{k=0}^{\alpha n+1} (-1)^k \binom{\alpha n+1}{k} \frac{1}{\binom{k-\frac{\alpha n+1}{\alpha}}{k}} = \frac{1}{1-\alpha}, \quad \alpha \geq 2 \quad (1.9)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^k k!}{\prod_{\alpha=0}^k (z+2\alpha)} = \frac{1}{z+2n}, \quad z \neq 0, -2, -4, \dots, -2n \quad (1.10)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{n-k} = \frac{1}{2} \binom{2n}{n} = \binom{2n-1}{n}, \quad n \geq 1 \quad (1.11)$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{1}{\binom{k+\frac{2n+1}{2}}{k}} = \frac{1}{3} \quad (1.12)$$

$$\sum_{k=0}^{2n+1} (-1)^k \frac{\binom{3n+2}{n+k+1}}{\binom{2n+2k+1}{n+k}} 2^{2k} = \frac{1}{3} \frac{\binom{3n+2}{n+1}}{\binom{2n+1}{n}} \quad (1.13)$$

1.2.2 Second Example: $f(x) \equiv x^n$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x-k)^n}{y+k} = \frac{(x+y)^n}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.14)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(n-k)^n}{n+k} = \frac{2^n n^{n-1}}{\binom{2n}{n}} \quad (1.15)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(n-k)^n}{1+k} = (n+1)^{n-1} \quad (1.16)$$

$$(x+1)^n = \sum_{\alpha=0}^n \frac{(-1)^\alpha}{\alpha+1} \sum_{k=\alpha}^n \binom{n}{k} \binom{k}{\alpha} (x-k)^{n-k} (k-\alpha)^k, \quad \text{with } 0^0 \equiv 1 \quad (1.17)$$

1.2.3 Third Example: $f(x) \equiv \binom{x+n}{n}$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x-k+n}{n} \frac{1}{y+k} = \frac{\binom{x+y+n}{n}}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.18)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \frac{1}{y+k} = \frac{\binom{2n+y}{n}}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.19)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{2n-k}{n}^2 \frac{1}{y+k} = \frac{\binom{2n}{n} \binom{2n+y}{n}}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.20)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{3n-k}{n} \frac{1}{2n-k} = \frac{(-1)^n}{n \binom{2n}{n}} \quad (1.21)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{n} \frac{1}{y+k} = \sum_{k=0}^n (-1)^k \binom{n+k}{2k} \binom{2k}{k} \frac{1}{y+k} = (-1)^n \frac{\binom{y-1}{n}}{y \binom{y+n}{n}} \quad (1.22)$$

1.2.4 Fourth Example: $f(x) = \binom{x^2}{n}$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{(y-k)^2}{n} \frac{1}{x+k} = \frac{\binom{(y+x)^2}{n}}{x \binom{x+2n}{2n}}, \quad x \neq 0, -1, -2, \dots, -2n \quad (1.23)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{k^2}{n} \frac{1}{x+k} = \frac{\binom{x^2}{n}}{x \binom{x+2n}{2n}}, \quad x \neq 0, -1, -2, \dots, -2n \quad (1.24)$$

1.2.5 Fifth Example: $f(x) = \binom{m x}{n}$

Remark 1.2 For this example, we assume m is a real or complex number.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m(x-k)}{n} \frac{1}{y+k} = \frac{\binom{m(x+y)}{n}}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.25)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{-mk}{n} \frac{1}{y+k} = \frac{\binom{my}{n}}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.26)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{mk}{n} \frac{1}{y+k} = (-1)^n \frac{\binom{-my+n-1}{n}}{y \binom{y+n}{n}}, \quad y \neq 0, -1, -2, \dots, -n \quad (1.27)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{z-yk}{n} \frac{z}{z-yk} = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases} \quad (1.28)$$

1.2.6 Sixth Example: $f(x) = \binom{z+\alpha x}{r} x^p$

Remark 1.3 In this example, we assume α is a real or complex number. We also assume r and p are positive integers.

$$\sum_{k=0}^{r+p} (-1)^k \binom{r+p}{k} \binom{z+\alpha x - \alpha k}{r} \frac{(x-k)^p}{y+k} = \frac{(x+y)^p}{y \binom{y+r+p}{r+p}} \binom{z+\alpha x + \alpha y}{r} \quad (1.29)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{z - \alpha k}{n} \frac{(x-k)^n}{y+k} = \frac{(x+y)^n}{y \binom{y+2n}{2n}} \binom{z + \alpha y}{n} \quad (1.30)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{z - \alpha k}{n} \frac{k^n}{k+1} = (-1)^n \frac{1}{2n+1} \binom{z + \alpha}{n} \quad (1.31)$$

1.2.7 Seventh Example: $f(x) = \binom{x}{r}^p$

Remark 1.4 For this example, we assume r and p are positive integers.

$$\sum_{k=0}^{rp} (-1)^k \binom{rp}{k} \binom{x-k}{r}^p \frac{1}{y+k} = \frac{\binom{x+y}{r}^p}{y \binom{y+rp}{n}}, \quad y \neq 0, -1, -2, \dots, -rp \quad (1.32)$$

$$\sum_{k=0}^{rp} (-1)^k \binom{rp}{k} \binom{rp-k}{r}^p \frac{1}{k+r} = \frac{1}{r} \binom{rp+r}{r}^{p-1} \quad (1.33)$$

1.3 Advanced Applications of Melzak's Formula

1.3.1 First Application: $f(x) \equiv 1$

$$\frac{1}{\binom{n+x}{n}} = \sum_{k=0}^n (-1)^k \frac{x}{x+k} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{x^{j+n}} \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} k^{j+n} \quad (1.34)$$

1.3.2 Second Application: $f(x) = x^n$

$$\sum_{k=0}^n \binom{n}{k} (x-k)^k (1-x+k)^{n-k} \frac{y}{y+k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x+y)^k}{\binom{y+k}{k}} \quad (1.35)$$

A Variation of Equation (1.35)

$$\sum_{k=0}^n \binom{n}{k} (x-bk)^k (b-x+bk)^{n-k} \frac{y}{y+k} = \sum_{k=0}^n (-1)^k \binom{n}{k} b^{n-k} \frac{(x+by)^k}{\binom{y+k}{k}}, \quad (1.36)$$

where b is a nonzero real or complex number.

Three Generalizations of Equation (1.35)

Remark 1.5 For the following three identities, we assume a and b are nonzero real or complex numbers. We also assume p is a nonnegative integer.

$$\sum_{k=0}^p \binom{p}{k} \frac{(by-bk)^k (a-by+bk)^{p-k}}{x+k} = \sum_{n=0}^p (-1)^n \binom{p}{n} a^{p-n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(by-bk)^n}{x+k} \quad (1.37)$$

$$\sum_{k=0}^p \binom{p}{k} \frac{(y-bk)^k (a-y+bk)^{p-k}}{x+k} = \sum_{n=0}^p (-1)^n \binom{p}{n} a^{p-n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(y-bk)^n}{x+k} \quad (1.38)$$

$$\sum_{k=0}^p \binom{p}{k} \frac{(y-bk)^k (1-y+bk)^{p-k}}{x+k} = \sum_{n=0}^p (-1)^n \binom{p}{n} \frac{(y+bx)^n}{x \binom{x+n}{n}} \quad (1.39)$$

1.3.3 Evaluation of $\sum_{k=1}^n \frac{1}{k}$

$$\sum_{k=1}^n \frac{1}{k} = \frac{f'(x)}{f(x)} - \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{f(x-k)}{k f(x)}, \quad n \geq 1 \quad (1.40)$$

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{f(x-k)}{k} = f(x) \sum_{k=1}^n \frac{1}{k} - f'(x), \quad n \geq 1 \quad (1.41)$$

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{x-k}{n} \frac{1}{k} = (x-n) \binom{x}{n} \sum_{k=1}^n \frac{1}{k(x+k-n)}, \quad n \geq 1 \quad (1.42)$$

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{n+k}{k} \frac{1}{k} = 2 \sum_{k=1}^n \frac{1}{k}, \quad n \geq 1 \quad (1.43)$$

1.3.4 Fourth Application: Let $f(x) = \frac{\binom{2x+2n}{2n}}{\binom{x+n}{n}}$

Remark 1.6 The following identity is the solution to Example III, (A), in “Théorie des Nombres” par Edouard Lucas, Tome Premier, Gauthier-Villars, Paris, Chapter XI, P. 187.

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{x+k} = \frac{\binom{2n}{n} \binom{2x+2n}{2n}}{x \binom{x+n}{n}^2} = \frac{2^{2n} \binom{n+x-\frac{1}{2}}{n}}{x \binom{x+n}{n}} \quad (1.44)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{x+k} = \frac{2^n (2x+1)(2x+3)\dots(2x+n-1)}{x(x+1)\dots(x+n)}, \quad n \geq 1 \quad (1.45)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k+1} = \frac{2n+1}{n+1} \binom{2n}{n} \quad (1.46)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{2k+1} = \frac{2^{4n}}{(2n+1) \binom{2n}{n}} = \frac{2^{2n}}{\binom{n+\frac{1}{2}}{n}} \quad (1.47)$$

1.3.5 Evaluation of $\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{b+k}{c}}$

Remark 1.7 The following identity is due to R. Frisch. It can be found in Eugen Netto's "Lehrbuch der Combinatorik", Leipzig, 1901, Sec. Ed. 1927, reprint by Chelsea Publ. Co., New York, N.Y. 1958. This identity can also be found in R. Frisch's "Sur les semi-invariants, etc.", Videnskaps-Akademiets Skrifter, II, No. 3, Oslo, 1926.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{b+k}{c}} = \frac{c}{n+c} \frac{1}{\binom{n+b}{b-c}}, \quad (1.48)$$

provided b and c are positive integers, $b \geq c$, and $\binom{b+k}{c} \neq 0$ for $k = 0, 1, 2, \dots, n$.

Two restatement of Equation (1.48)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{c}{k+c} \frac{1}{\binom{k+b}{b-c}} = \frac{1}{\binom{b+n}{c}} \quad (1.49)$$

provided b and c are positive integers, $b \geq c$, and $\binom{b+k}{b-c} \neq 0$ for $k = 0, 1, 2, \dots, n$.

$$\sum_{k=0}^{\alpha n} (-1)^k \binom{\alpha n}{k} \frac{1}{\binom{\beta n+k}{k}} = \frac{\beta}{\alpha + \beta}, \quad \alpha \text{ and } \beta \text{ positive integers, } \alpha \neq \beta. \quad (1.50)$$

Specific Example of Equation (1.48)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{\binom{3n+k}{2n}} = \frac{2}{3 \binom{4n}{n}} \quad (1.51)$$

1.4 Generalized Melzak's Formula

Remark 1.8 In the following identity, we evaluate $x! = \Gamma(x+1)$.

$$\sum_{k=0}^{\infty} (-1)^k \binom{x}{k} \frac{1}{z+k} = \frac{\Gamma(z)\Gamma(x+1)}{\Gamma(x+z+1)} = \frac{1}{z \binom{x+z}{x}}, \quad (1.52)$$

where $Re(x) > -1$ and $z \neq 0, -1, -2, \dots$

1.4.1 Applications of Equation (1.52)

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}} \frac{1}{z+k} = \frac{1}{z \binom{x-\frac{1}{2}}{z}} \quad (1.53)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}(z+2k+1)} = \frac{\pi}{2^{z+1}} \binom{z}{\frac{z}{2}} \quad (1.54)$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}(2k+1)} = \frac{\pi}{2} \quad (1.55)$$

1.5 Derivative Version of Melzak's Formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(x-k)}{(y+k)^2} = \frac{f(x+y) \sum_{k=0}^n \frac{1}{k+y} - \frac{d}{dy} f(x+y)}{y \binom{y+n}{n}}, \quad (1.56)$$

$y \neq 0, -1, \dots, -n$

1.5.1 Applications of Equation (1.56)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x-k)^n}{(y+k)^2} = \frac{(x+y)^n \sum_{k=0}^n \frac{1}{k+y} - n(x+y)^{n-1}}{y \binom{y+n}{n}} \quad (1.57)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k^n}{(1+k)^2} = (-1)^n \frac{\sum_{k=1}^{n+1} \frac{1}{k} - n}{n+1} \quad (1.58)$$

$$\sum_{k=0}^n \frac{1}{k+y} = \frac{y \binom{y+n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x-k)^n}{(y+k)^2} + n(x+y)^{n-1}}{(x+y)^n}, \quad y \neq 0, -1, \dots, -n \quad (1.59)$$

$$\sum_{k=1}^{n+1} \frac{1}{k} = (-1)^n (n+1) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k^n}{(k+1)^2} + n, \quad 0^0 \equiv 1 \quad (1.60)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{(y+k)^2} = \frac{1}{y \binom{y+n}{n}} \sum_{k=0}^n \frac{1}{k+y} \quad (1.61)$$

1.6 Melzak's Formula for Polynomials of Degree $n + 1$

Remark 1.9 In this section, we assume $f(x)$ is a polynomial of degree $n + 1$, that is, $f(x) = \sum_{i=0}^{n+1} a_i x^i$.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k} = \frac{f(x+y)}{y \binom{y+n}{n}} - n! a_{n+1}, \quad (1.62)$$

$$y \neq 0, -1, \dots, -n, \quad n \geq 1$$

1.6.1 Examples of Equation (1.62)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x-k)^{n+1}}{y+k} = \frac{(x+y)^{n+1}}{y \binom{y+n}{n}} - n!, \quad n \geq 1 \quad (1.63)$$

$$\sum_{k=0}^{rp-1} (-1)^k \binom{rp-1}{k} \binom{x-k}{r}^p \frac{1}{y+k} = \frac{\binom{x+y}{r}^p}{y \binom{y+rp-1}{y}} - \frac{(rp-1)!}{r!^p}, \quad (1.64)$$

r and p are positive integers, $rp \geq 2$, $y \neq 0, -1, -2, \dots, 1 - rp$

1.7 Partial Fraction Generalizations of Melzak's Formula

1.7.1 First Partial Fraction Decomposition

Let $f(x) = \sum_{i=0}^{n+1} a_i x^i$. Then,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(y-k)}{(k+x)(k+z)} = \frac{1}{(n+1)(x-z)} \left(\frac{f(y+z)}{\binom{z+n}{n+1}} - \frac{f(y+x)}{\binom{x+n}{n+1}} \right), \quad x \neq z. \quad (1.65)$$

1.7.2 Second Partial Fraction Decomposition

Assume u, v , and w are real or complex numbers. Let $f(x) = \sum_{k=0}^{n+2} a_k x^k$. Then,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(y-k)}{(k+u)(k+v)(k+w)} = \frac{u-1}{(n+1)(n+2)(u-v)(u-w)} \left(\frac{f(y+u)}{\binom{u+n}{n+2}} - \frac{f(y+w)}{\binom{w+n}{n+2}} \right)$$

$$- \frac{v-1}{(n+1)(n+2)(u-v)(v-w)} \left(\frac{f(y+v)}{\binom{v+n}{n+2}} - \frac{f(y+w)}{\binom{w+n}{n+2}} \right), \quad u \neq v \neq w. \quad (1.66)$$

Example of Equation (1.66)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(n+1)(n+2)}{(k+3)(k+4)(k+7)} = \frac{1}{2(n+3)} - \frac{1}{\binom{n+4}{2}} + \frac{1}{2\binom{n+7}{5}} \quad (1.67)$$

1.7.3 Third Partial Fraction Decomposition

Let $f(x) = \sum_{i=0}^{n+\alpha-1} a_i x^i$. Then,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(y-k)}{\binom{k+\alpha}{k}} = - \sum_{k=1}^{\alpha} (-1)^k \binom{\alpha}{k} \frac{(y+k)}{\binom{k+n}{k}}, \quad \alpha \geq 1. \quad (1.68)$$

Two Applications of Equation (1.68)

First Application

Let $f(x) = \sum_{i=0}^{2n-1} a_i x^i$. Then,

$$f(y) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(y-k) + f(y+k)}{\binom{k+n}{k}}. \quad (1.69)$$

Second Application

Let $f(x) = \sum_{i=1}^p a_i x^{2i}$ and assume $1 \leq p \leq n-1$. Then,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(k)}{\binom{k+n}{n}} = 0. \quad (1.70)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k^{2p}}{\binom{k+n}{n}} = 0, \text{ provided } p \text{ is a positive integer, } 1 < 2p < 2n-1 \quad (1.71)$$

2 Lagrange Interpolation Formula

Remark 2.1 Throughout this chapter, we assume, unless otherwise specified, that n and j are nonnegative integers, and x , y , and z are real or complex numbers.

2.1 Two Basic Forms of the Lagrange Interpolation Formula

2.1.1 Product Form

Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Then, for $n \geq 1$,

$$f(x) = \sum_{k=0}^n f(x_k) \prod_{\substack{i=0 \\ i \neq k}}^n \frac{x - x_i}{x_k - x_i}, \text{ where the } x_i \text{ are distinct.} \quad (2.1)$$

2.1.2 Derivative Form

Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Let $\varphi(x) = \prod_{i=0}^n (x - x_i)$. Then, for $n \geq 1$,

$$f(x) = \varphi(x) \sum_{k=0}^n \frac{f(x_k)}{(x - x_k)\varphi'(x_k)}, \text{ where the } x_i \text{ are distinct.} \quad (2.2)$$

Generalization of Equation (2.2)

Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Let $\varphi(x) = \prod_{i=0}^n (x - x_i)$. Then, for $n \geq 1$,

$$f(x + y) = \varphi(x) \sum_{k=0}^n \frac{f(y + x_k)}{(x - x_k)\varphi'(x_k)}, \text{ where the } x_i \text{ are distinct.} \quad (2.3)$$

Integral Form of Equation (2.2)

Assume x is a real number. Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Let $\varphi(x) = \prod_{i=0}^n (x - x_i)$. Then, for $n \geq 1$,

$$\int \frac{f(x)}{\varphi(x)} dx = \sum_{k=0}^n \frac{f(x_k)}{\varphi'(x_k)} \ln(x - x_k) + C, \text{ where the } x_i \text{ are distinct.} \quad (2.4)$$

Derivative Form of Equation (2.2)

Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Let $\varphi(x) = \prod_{i=0}^n (x - x_i)$. Then, for $n \geq 1$,

$$\left(\frac{f(x)}{\varphi(x)} \right)' = (-1)^n n! \sum_{k=0}^n \frac{f(x_k)}{(x - x_k)^{n+1} \varphi'(x_k)}, \text{ where the } x_i \text{ are distinct.} \quad (2.5)$$

2.2 Partial Fraction Expansion Via Equation (2.2) and Melzak's Formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k^m}{\prod_{i=1}^{\alpha} (k - x_i)} = - \sum_{j=1}^{\alpha} \frac{x_j^{m-1}}{\binom{n-x_j}{n} \prod_{\substack{i=1 \\ i \neq j}}^{\alpha} (x_j - x_i)}, \text{ } m \text{ a positive integer} \quad (2.6)$$

2.3 Applications of Equation (2.3)

2.3.1 First Application: Let $x_i = i$

$$f(x + y) = \sum_{k=0}^n f(k + y) \prod_{\substack{j=0 \\ j \neq k}}^n \frac{x - j}{k - j} = \sum_{k=0}^n \binom{x}{k} \binom{n-x}{n-k} f(k + y) \quad (2.7)$$

$$\sum_{k=0}^n \binom{x}{k} \binom{n-x}{n-k} \binom{k+y}{n} = \binom{x+y}{n} \quad (2.8)$$

2.3.2 Second Application: Let $x_i = i^2$

Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Then, for $n \geq 1$

$$\sum_{k=0}^n (-1)^k \binom{2n}{n+k} \frac{f(y+k^2)}{x^2-k^2} = \frac{(-1)^n f(x^2+y)}{2x(x-n)\binom{x+n}{2n}} + \frac{1}{2} \binom{2n}{n} \frac{f(y)}{x^2}. \quad (2.9)$$

Let $f(x)$ be a polynomial of degree n , that is, $f(x) = \sum_{i=0}^n a_i x^i$. Then, for $n \geq 1$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(y+k^2)}{(x^2-k^2)\binom{n+k}{k}} = \frac{(-1)^n f(x^2+y)}{2x(x-n)\binom{x+n}{2n}\binom{2n}{n}} + \frac{f(y)}{2x^2}. \quad (2.10)$$

Specific Cases of Equation (2.9)

$$\sum_{k=0}^n (-1)^k \binom{2n}{n+k} = \binom{2n-1}{n}, \quad n \geq 1 \quad (2.11)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{n+k} \frac{1}{x^2-k^2} = \frac{1}{2x^2} \binom{2n}{n} + \frac{1}{2x(x-n)\binom{x+n}{2n}} \quad (2.12)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{n+k} \frac{1}{n^3-k^2} = \frac{1}{2n^3} \binom{2n}{n} + \frac{(-1)^n}{2n^2 \sqrt{n}(\sqrt{n}-1)\binom{n+n\sqrt{n}}{2n}}, \quad n \geq 2 \quad (2.13)$$

2.3.3 Third Application: Roots of Unity

Remark 2.2 Throughout this subsection, we define w_n^α , for $\alpha = 1, 2, \dots, n$, to be an n^{th} root of unity, i.e. $w_n = e^{\frac{2\pi i}{n}}$, where $i = \sqrt{-1}$. We will also assume $f(x)$ is a polynomial of degree $n-1$, namely, $f(x) = \sum_{j=0}^{n-1} a_j x^j$.

$$\frac{f(x)}{x^n-1} = \frac{1}{n} \sum_{k=1}^n \frac{f(w_n^k) w_n^k}{x-w_n^k}, \quad n \geq 1 \quad (2.14)$$

$$\frac{x^r}{x^n-1} = \frac{1}{n} \sum_{k=1}^n \frac{(w_n^k)^{r+1}}{x-w_n^k}, \quad n \geq 1, \quad r \text{ an integer}, \quad 0 \leq r \leq n-1 \quad (2.15)$$

$$\frac{1}{x^n-1} = \frac{1}{n} \sum_{k=1}^n \frac{w_n^k}{x-w_n^k}, \quad n \geq 1 \quad (2.16)$$

2.3.4 Fourth Application: Shifted Roots of Unity

Remark 2.3 Throughout this subsection, we define w_n^α , for $\alpha = 1, 2, \dots, n$, to be an n^{th} root of $z^n = -1$, i.e. $w_n = e^{\frac{2k+1}{n}\pi i}$, where $i = \sqrt{-1}$. We will also assume $f(x)$ is a polynomial of degree $n - 1$, namely, $f(x) = \sum_{j=0}^{n-1} a_j x^j$.

$$\frac{f(x)}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^n \frac{f(w_n^k) w_n^k}{x - w_n^k}, \quad n \geq 1 \quad (2.17)$$

$$\frac{x^r}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^n \frac{(w_n^k)^{r+1}}{x - w_n^k}, \quad n \geq 1, \quad r \text{ an integer}, \quad 0 \leq r \leq n - 1 \quad (2.18)$$

$$\frac{1}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^n \frac{w_n^k}{x - w_n^k}, \quad n \geq 1 \quad (2.19)$$

3 Hagen/Rothe Coefficients

Remark 3.1 Throughout this chapter, we assume n , α , and β are nonnegative integers. We also assume x , y , and z are real or complex numbers. For real x , we let $[x]$ denote the floor of x .

Remark 3.2 There are two main references for the material in this Chapter. The first is Johann Georg Hagen's *Synopsis der hoeheren Mathematik*, Berlin, 1891 (3 Volumes). The second is Heinrich August Rothe's, *Formulae de serierum reversione*, etc. Leipzig, 1793.

3.1 Hagen/Rothe Coefficients of the First Type

We let Hagen/Rothe Coefficient of the First Type be denoted by $A_n(x, d)$, where,

$$A_n(x, z) = \frac{x}{x + nz} \binom{x + nz}{n}. \quad (3.1)$$

Alternative Definition of $A_n(x, z)$

$$A_n(x, z) = \sum_{k=0}^{n-1} (-1)^{k+n+1} \binom{n}{k} \binom{x + kz}{n} \frac{x}{x + kz}, \quad n \geq 1 \quad (3.2)$$

3.1.1 Generating Function for $A_k(\alpha, \beta)$

$$\sum_{k=0}^{\infty} A_k(\alpha, \beta) z^k = y^\alpha, \text{ where } z = \frac{y-1}{y^\beta}, \text{ and } |z| < \left| \frac{(\beta-1)^{\beta-1}}{\beta^\beta} \right| \quad (3.3)$$

3.1.2 Properties of $A_n(x, z)$

n^{th} Difference Property

$$\sum_{k=0}^n (-1)^k \binom{n}{k} A_n(k, z) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+nz}{n} \frac{k}{k+nz} = (-1)^n, \quad n \geq 1 \quad (3.4)$$

Convolution Formula

$$\sum_{k=0}^n A_k(x, z) A_{n-k}(y, z) = A_n(x+y, z), \quad i.e. \quad (3.5)$$

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y+nz-kz}{n-k} \frac{x}{x+kz} \cdot \frac{y}{y+nz-kz} = \binom{x+y+nz}{n} \frac{x+y}{x+y+nz} \quad (3.6)$$

Generalization of Equation (3.5): Let p and q be real or complex numbers. Then,

$$\sum_{k=0}^n A_k(x, z) A_{n-k}(y, z) (p+qk) = \frac{p(x+y) + nxq}{x+y+zn} \binom{x+y+zn}{n}, \quad (3.7)$$

Hagen Identity

$$\sum_{k=0}^n \binom{x+zk}{k} \binom{y-zk}{n-k} \frac{p+qk}{(x+zk)(y-zk)} = \frac{p(z+y-zn) + nxq}{(x+y)x(y-zn)} \binom{x+y}{n} \quad (3.8)$$

Vosmansky Extension

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} A_{2n}(k, x) A_{2n}(2n-k, x) = (-1)^n x \binom{2n}{n} A_{2n}(n, x), \quad i.e. \quad (3.9)$$

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{k+2nx}{2n} \binom{2n-k+2nx}{2n} \frac{k(2n-k)}{(k+2nx)(2n-k+2nx)} = \\ (-1)^n \binom{2n}{n} \binom{n+2nx}{2n} \frac{x}{1+2x} = \frac{(-1)^n}{2 \cdot n!^2} \prod_{j=0}^{n-1} ((2nx)^2 - j^2), \quad n \geq 1 \end{aligned} \quad (3.10)$$

Restatement of Equation (3.9)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} A_n(k, x) A_n(n-k, x) = (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor + nx}{n} \frac{\lfloor \frac{n}{2} \rfloor x}{\lfloor \frac{n}{2} \rfloor + nx} \frac{(-1)^n + 1}{2} \quad (3.11)$$

3.2 Hagen/Rothe Coefficients of the Second Type

Remark 3.3 To form the Hagen/Rothe Coefficient of the Second Type, simply delete the fraction factor of $A_n(x, z)$.

3.2.1 Generating Function for Hagen/Rothe Coefficients of the Second Type

$$\sum_{k=0}^{\infty} \binom{\alpha + \beta k}{k} z^k = \frac{y^{\alpha+1}}{(1-\beta)y + \beta}, \text{ where } z = \frac{y-1}{y^\beta}, \text{ and } |z| < \left| \frac{(\beta-1)^{\beta-1}}{\beta^\beta} \right| \quad (3.12)$$

3.2.2 Properties of Hagen/Rothe Coefficients of the Second Type

n^{th} Difference Formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+yk}{n} = (-1)^n y^n \quad (3.13)$$

First Convolution Formula

$$\sum_{k=0}^n \binom{x+k\beta}{k} \binom{y+(n-k)\beta}{n-k} = \sum_{k=0}^n \binom{x+z+k\beta}{k} \binom{y-z+(n-k)\beta}{n-k} \quad (3.14)$$

Second Convolution Formula

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y+(n-k)z}{n-k} \frac{x}{x+kz} = \binom{x+y+nz}{n} \quad (3.15)$$

Variation of Equation (3.15)

$$\sum_{k=0}^n \binom{x+kz}{z} \binom{y-kz}{n-k} \frac{1}{x+kz} = \frac{1}{x} \binom{x+y}{n} \quad (3.16)$$

Remark 3.4 The following identity, proposed by K. L. Chung, is from Problem 4211 of *The American Math. Monthly*, 1946, P. 397. Assume d is a real or complex number. Then,

$$\sum_{k=1}^n \binom{dk}{k} \binom{nd - kd}{n - k} \frac{1}{dk - 1} = \frac{1}{d - 1} \binom{nd}{n}, \quad n \geq 1, \quad d \neq 1. \quad (3.17)$$

Variation of Equation (3.17)

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{(d-1)n}{(d-1)k}}{\binom{dn}{dk}} \frac{1}{dk - 1} = \frac{d - 2}{1 - d}, \quad d \neq 1 \quad (3.18)$$

3.3 Abel Coefficients

Remark 3.5 In this section, we define the Abel Coefficients, $B(x, z)$, as

$$B_n(x, z) = \frac{x}{x + nz} \frac{(x + nz)^n}{n!}. \quad (3.19)$$

The Abel Coefficients are obtained from $A_n(x, z)$ by a simple limiting process. A reference for the material in this subsection is N. H. Abel's, "Demonstration d'une expression de laquelle la formule binôme est un cas particulier", *Jour. reine u. angew. Math.*, Vol. 1 (1826), P. 159.

3.3.1 Generating Functions for $B_k(\alpha, \beta)$

$$\sum_{k=0}^{\infty} B_k(\alpha, \beta) z^k = x^\alpha = e^{\alpha w} : x = e^w, z = we^{-\beta w}, |\beta we^{-\beta w}| < \frac{1}{e} \quad (3.20)$$

$$\sum_{k=0}^{\infty} B_k(\alpha, \beta) z^k \frac{\alpha + \beta k}{\alpha} = \frac{x^\alpha}{1 - \beta \ln x} = \frac{e^{\alpha w}}{1 - \beta w} : x = e^w, z = we^{-\beta w}, |\beta we^{-\beta w}| < \frac{1}{e} \quad (3.21)$$

3.3.2 Properties of $B_n(x, z)$

Remark 3.6 For the following three identities, we assume p, q , and t are real or complex numbers.

$$\sum_{k=0}^n B_k(x, z) B_{n-k}(y, z) = B_n(x + y, z) \quad (3.22)$$

$$\sum_{k=0}^n B_k(x, z) B_{n-k}(y, z) (p + qk) = \frac{p(x+y) + qnx}{x+y} B_n(x+y, z) \quad (3.23)$$

$$\begin{aligned} \sum_{k=0}^n B_k(x, \beta) \frac{x + \beta k}{x} B_{n-k}(y, \beta) \frac{y + \beta(n-k)}{y} = \\ \sum_{k=0}^n B_k(x+t, \beta) \frac{x+t+\beta k}{x+t} B_{n-k}(y-t, \beta) \frac{y-t+\beta(n-k)}{y-t} \end{aligned} \quad (3.24)$$

3.4 Generalization of $A_n(x, z)$

Remark 3.7 In this section, we assume w is a real or complex number.

$$\begin{aligned} \frac{w}{w+zn} \binom{x+zn}{n} &= (-1)^n \frac{\binom{x-w}{n}}{\binom{\frac{w}{z}+n}{n}} \\ &+ \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{n}{k} \binom{x+zk}{n} \frac{w}{w+zk}, \quad n \geq 1, z \neq 0 \end{aligned} \quad (3.25)$$

Generating Function

$$\sum_{k=0}^{\infty} \binom{\alpha + \beta k}{k} \frac{w}{w + \beta k} z^k = x^\alpha \sum_{n=0}^{\infty} (-1)^n \frac{\binom{\alpha-w}{n}}{\binom{\frac{w}{\beta}+n}{n}} \left(\frac{x-1}{x}\right)^n, \quad z = \frac{x-1}{x^\beta} \quad (3.26)$$

Remark 3.8 In the following equation, assume

$$F(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{\alpha + \beta k}{n} f(k), \quad f(0) = 1. \quad (3.27)$$

Then,

$$\sum_{k=0}^{\infty} \binom{\alpha + \beta k}{k} z^k f(k) = x^\alpha \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-1}{x}\right)^n F(n), \quad z = \frac{x-1}{x^\beta} \quad (3.28)$$

3.5 Generalized Rothe Convolutions

Remark 3.9 In this section, given a function $f(x)$, we assume there exists coefficients $C_k(\alpha, \beta)$ such that

$$x^\alpha = \sum_{k=0}^{\infty} C_k(\alpha, \beta) z^k, \quad \text{where } z = \frac{f(x)}{x^\beta}. \quad (3.29)$$

Using convolution theory, we show that

$$\sum_{k=0}^n C_k(\alpha, \beta) C_{n-k}(\gamma, \beta) = C_n(\alpha + \gamma, \beta). \quad (3.30)$$

A reference for this material is J. G. Van der Corput's Over eenige identiteiten, *Nieuw Archief voor Wiskunde, Series 2, Vol. 17, 1931, pp. 17-27.*

3.5.1 Applications of Equation (3.30)

$$\sum_{k=1}^{n-1} \binom{\beta k}{k} \binom{\beta n - \beta k}{n - k} \frac{1}{k(n - k)} = \frac{2\beta}{n} \binom{\beta n}{n} \sum_{k=1}^{n-1} \frac{1}{\beta n - n + k}, \quad n \geq 2 \quad (3.31)$$

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1} (n - k)^{n-k-1} = 2(n - 1)n^{n-2}, \quad n \geq 2 \quad (3.32)$$

3.6 Generalized Difference Series

3.6.1 Differential Series

Remark 3.10 Throughout this subsection, we use the convention that $D_x^n f(x)$ is the n^{th} derivative with respect to x .

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} [D_t^n f(t)]_{t=0} \quad (3.33)$$

Abel Series

$$f(x) = \sum_{n=0}^{\infty} \frac{x(x - n)^{n-1}}{n!} [D_t^n f(t)]_{t=n} \quad (3.34)$$

Generalized Differential Series

$$f(x) = \sum_{n=0}^{\infty} \frac{x}{x + yn} \frac{(x + yn)^n}{n!} [D_t^n f(t)]_{t=-yn} \quad (3.35)$$

3.6.2 Difference Series

Remark 3.11 Throughout this section, we let $\Delta_{t,1}^n f(t)$ denote the n^{th} difference of $f(t)$, namely,

$$\Delta_{t,1}^n f(t) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} f(x+k). \quad (3.36)$$

Remark 3.12 The material in this section can be found in N. E. Nörlund's *Vorlesungen über Differenzenrechnung*, Berlin, 1924. (Reprinted by Chelsea Publ. Co., New York, N. Y., 1954.) Also see N. Nielsen's *Handbuch der Theorie der Gammafunktion*, Leipzig, 1906. We also refer the reader to R. C. Buck's "Interpolation Series", *Trans. Amer. Math. Soc.*, Vol. 64, 1948, pp 283-298.

Newton-Gregory Series

$$f(x) = \sum_{n=0}^{\infty} \binom{x}{n} [\Delta_{t,1}^n f(t)]_{t=0} \quad (3.37)$$

Stirling Series

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x}{n} \binom{x-1+\frac{n}{2}}{n-1} [\Delta_{t,1}^n f(t)]_{t=-\frac{n}{2}} \quad (3.38)$$

Generalized Difference Series: Two Versions

$$f(x) = \sum_{n=0}^{\infty} \frac{x}{x+yn} \binom{x+yn}{n} [\Delta_{t,1}^n f(t)]_{t=-yn} \quad (3.39)$$

$$f(x+y) = \sum_{n=0}^{\infty} \binom{\frac{x+\beta n}{z}}{n} \frac{x}{x+\beta n} z^n [\Delta_{t,z}^n f(t+y)]_{t=-\beta n} \quad (3.40)$$

4 Jensen's Formula

Remark 4.1 In this chapter, we assume u, v, α, β , and γ , are real or complex numbers, while n and p are nonnegative integers. We also note that a different variation of Equation (4.1) is found in J. L. W. V. Jensen's "Sur une identité d'Abel et sur d'autres formules analogues", *Acta Mathematica*, Vol. 26, 1902, pp. 307-318.

Jensen's Formula

$$\sum_{k=0}^n \binom{\alpha + \beta k}{k} \binom{\gamma - \beta k}{n-k} = \sum_{k=0}^n \binom{\alpha + \gamma - k}{n-k} \beta^k \quad (4.1)$$

4.1 Specific Evaluations of Equation (4.1)

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} = 2^{2n} \quad (4.2)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{n-2k}{n-k} = 2^{n+1} - 1 \quad (4.3)$$

$$\sum_{k=0}^n \binom{\alpha + \beta k}{k} \binom{p - \alpha - \beta k}{n - k} = \beta^{p+1} (\beta - 1)^{n-p-1}, \quad 0 \leq p \leq n - 1 \quad (4.4)$$

$$\sum_{k=0}^n \binom{\alpha + \beta k}{k} \binom{n - \alpha - \beta k}{n - k} = \frac{\beta^{n+1} - 1}{\beta - 1}, \quad \beta \neq 1 \quad (4.5)$$

$$v^n \sum_{k=0}^n \binom{\frac{u}{v}k}{k} \binom{n - \frac{u}{v}k}{n - k} = \frac{u^{n+1} - v^{n+1}}{u - v}, \quad v \neq 0, \quad u \neq v \quad (4.6)$$

$$F_{n+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} = \left(\frac{1+\sqrt{5}}{2}\right)^n \sum_{k=0}^n \binom{\frac{1+\sqrt{5}}{2}k}{k} \binom{n - \frac{1+\sqrt{5}}{2}k}{n - k}, \quad (4.7)$$

where $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}$ are the Fibonacci Numbers

4.2 Limiting Case of Equation (4.1)

$$\sum_{k=0}^n \frac{(\alpha + \beta k)^k (\gamma - \beta k)^{n-k}}{k! (n - k)!} = \beta^n \sum_{k=0}^n \frac{(\alpha + \gamma)^k}{\beta^k k!} \quad (4.8)$$

5 Lagrange Inversion Theorem

Remark 5.1 Throughout this chapter, we assume t , h , x , y , and z are real or complex numbers, while n is a nonnegative integer. We also let $D_t^n f(t)$ denote the n^{th} derivative of $f(t)$ with respect to t .

5.1 Basic Statement of the Lagrange Inversion Theorem

Let $x = t + h\varphi(x)$. Then,

$$f(x) = \sum_{n=0}^{\infty} \frac{h^n}{n!} D_t^{n-1} (f'(t)(\varphi(t))^n). \quad (5.1)$$

Remark 5.2 For an analytic proof of the Lagrange Inversion Theorem, also known as Bürmann's Theorem, we refer the reader to three sources. The first is Modern Analysis, Fourth Edition, by Whittaker and Watson, 1927, P. 133. We also suggest

An Introduction to the Theory of Functions of a Complex Variable by E. T. Copson, Oxford University Press, First Ed. 1935, reprint 1944, pp. 123-125. The third source is T. W. Chaundy's "The Validity of Lagrange's Expansion", Quarterly Journal of Mathematics, Oxford Series, Vol. 2, 1931, pp. 284-297.

5.2 Derivative Formulas Associated with the Lagrange Inversion Theorem

Remark 5.3 The formulas in this section are given as exercises in Recueil d'Exercices sur le Calcul Infinitésimal, Second Edition, by Frederic Frenet, Paris, 1866, Page 15 and Pages 75-76.

Let $z = x + yf(x)$. Let $D_x^n f(x)$ denote the n^{th} derivative of $f(x)$ with respect to x . Furthermore, assume $F(z)$ is any continuous, differentiable function. Let $z = x + yf(x)$. Then,

$$D_y z = f(z) D_x z \quad (5.2)$$

$$D_x z = \frac{1}{1 - yf'(z)} \quad (5.3)$$

$$D_y^n z = D_x^{n-1} ((f(x))^n D_x z) \quad (5.4)$$

Duhamel's Theorem

$$D_y^n F(z) = D_x^{n-1} (F'(z)(f(x))^n D_x z) \quad (5.5)$$

Lagrange Inversion Formula

$$F(z) = \sum_{n=0}^{\infty} \frac{y^n}{n!} D_x^{n-1} (F'(x)(f(x))^n) \quad (5.6)$$

6 Basic Operators

Remark 6.1 Throughout this chapter, we assume n and p are nonnegative integers. We also assume x is a real or complex number, while h is a nonzero real or complex number. We also let D_x denote the derivative with respect to x .

6.1 The Operator $(x\Delta_h)^n$

Remark 6.2 Recall that

$$\Delta_h f(x) = \frac{f(x+h) - f(x)}{h}, \quad (6.1)$$

is the (first) difference operator.

$$(x\Delta_h)^n f(x) = \sum_{j=0}^n \binom{j + \frac{x}{h} - 1}{j} h^j \Delta_h^j f(x) \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n \quad (6.2)$$

6.1.1 Specific Evaluations of Equation (6.2)

$$(x\Delta_1)^n f(x) = \sum_{j=0}^n \binom{j+x-1}{j} \Delta_1^j f(x) \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n \quad (6.3)$$

$$(x\Delta_{\frac{1}{2}})^n f(x) = \sum_{j=0}^n \binom{j+2x-1}{j} \frac{1}{2^j} \Delta_{\frac{1}{2}}^j f(x) \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n \quad (6.4)$$

$$(x\Delta_2)^n f(x) = \sum_{j=0}^n \binom{j+\frac{x}{2}-1}{j} 2^j \Delta_2^j f(x) \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n \quad (6.5)$$

$$(x\Delta_h)^n x^p = \sum_{j=0}^n \binom{\frac{x}{h} + j - 1}{j} B_{j,j}^n \sum_{r=0}^p \binom{p}{r} x^{p-r} h^r B_{j,j}^r, \text{ where} \quad (6.6)$$

$$B_{j,j}^n = \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n$$

$$(x\Delta_1)^n \binom{x}{p} = \sum_{j=0}^n \binom{x+j-1}{j} \binom{x}{p-j} \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n \quad (6.7)$$

6.2 The Operator $(xE_h)^n$

Remark 6.3 Recall that E_h is the shift operator, that is $E_h f(x) = f(x + h)$.

$$(xE_h)^n f(x) = h^n n! \binom{\frac{x}{h} + n - 1}{n} f(x + nh) = h^n n! \binom{\frac{x}{h} + n - 1}{n} E_h^n f(x) \quad (6.8)$$

6.3 The Operator $(x^r D_x)^n$

6.3.1 The Operator $(xD_x)^n$

Grunert's Formula

Let S be function of x . Then,

$$(xD_x)^n S = \sum_{j=0}^n \sum_{k=0}^j \frac{(-1)^{j+k}}{j!} \binom{j}{k} k^n x^j D_x^j S. \quad (6.9)$$

6.3.2 The Operator $(x^2 D_x)^n$

Let S be function of x . Then,

$$(x^2 D_x)^n S = \sum_{j=0}^n \frac{n!}{j!} \binom{n-1}{j-1} x^{n+j} D_x^j S. \quad (6.10)$$

6.3.3 The Operator $(\frac{1}{x} D_x)^n$

Let S be function of x . Then,

$$\left(\frac{1}{x} D_x\right)^n S = \sum_{k=0}^n (-1)^{n-k} \frac{kn!}{(2n-k)k!} \binom{2n-k}{n} \frac{x^{k-2n}}{2^{n-k}} D_x^k S. \quad (6.11)$$

6.3.4 Recursive Formula for $(x^r D_x)^n$

Let S be a function of x . Let r be any real number. Define $T_{r,0}^0 \equiv 1$. Assume $T_{r,0}^n \equiv 0$, for $n \geq 1$; Also define $T_{r,j}^n \equiv 0$ for $j > n$. Then,

$$(x^r D_x)^n S = \sum_{j=0}^n T_{r,j}^n x^{(r-1)n+j} D_x^j S, \text{ where } T_{r,j+1}^{n+1} = ((r-1)n + j + 1) T_{r,j+1}^n + T_{r,j}^n. \quad (6.12)$$

Specific Evaluations of Equation (6.12)

$$\sum_{k=0}^n \frac{p!}{(p-k)!} T_{r+1,k}^n = r^n n! \binom{\frac{p}{r} + n - 1}{n} \quad (6.13)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{p}{k} k = n \binom{p+n-1}{n}, \text{ where } p \text{ is a real or complex number} \quad (6.14)$$

Generalization of Equation (6.13)

$$\sum_{k=0}^n \binom{x}{k} k! T_{r+1,k}^n = r^n n! \binom{\frac{x}{r} + n - 1}{n} \quad (6.15)$$

Specific Evaluations of Equation (6.15)

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{2n-k}{n} \frac{2^k k}{2n-k} = 2^{2n} \binom{\frac{-x}{2} + n - 1}{n} \quad (6.16)$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} \binom{x}{n-k} \frac{n-k}{2^k(n+k)} = 2^n \binom{\frac{x}{2}}{n} \quad (6.17)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-k}{n} \frac{k}{2^k(2n-k)} = (-1)^n 2^{2n} \binom{\frac{-1}{4}}{n} \quad (6.18)$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} \binom{x+1}{n-k} 2^{-n-k} = \binom{\frac{x}{2}}{n} \quad (6.19)$$

6.4 The Operator $(D_x x^r)^n$

6.4.1 Operator $(D_x x)^n$

Let S be function of x . Then,

$$(D_x x)^n S = \sum_{j=0}^n \sum_{k=0}^{j+1} \frac{(-1)^{k+j+1}}{(j+1)!} \binom{j+1}{k} k^{n+1} x^j D_x^j S. \quad (6.20)$$

6.4.2 Recursive Formula for $(D_x x^r)^n$

Let S be a function of x . Let r be any real number. Define $T_{r,0}^0 \equiv 1$. Assume $T_{r,0}^n \equiv 0$, for $n \geq 1$. Also define $T_{r,j}^n \equiv 0$ for $j > n$. Then,

$$(D_x x^r)^n S = \sum_{j=0}^n T_{r,j+1}^{n+1} x^{(r-1)n+j} D_x^j S, \text{ where } T_{r,j+1}^{n+1} = ((r-1)n + j + 1)T_{r,j+1}^n + T_{r,j}^n. \quad (6.21)$$

6.4.3 Combined with $x^r D_x$

$$(D_x x - x D_x)^n f(x) = f(x) \quad (6.22)$$

$$(D_x x^r - x^r D_x)^n f(x) = r^n x^{(r-1)n} f(x) \quad (6.23)$$

6.5 The Operator $(x D_x x)^n$

Let S be a function of x . Then,

$$(x D_x x)^n S = \sum_{k=0}^n \frac{n!}{k!} \binom{n}{k} x^{k+n} D_x^k S. \quad (6.24)$$

6.6 The Operator $(x D_x^p)^n$

Let S be a function of x . Let r be any real number. Define $T_{r,0}^0 \equiv 1$; $T_{r,0}^n \equiv 0$, for $n \geq 1$; $T_{r,j}^n \equiv 0$ for $j > n$. Then,

$$(x D_x^p)^n S = \sum_{j=0}^n T_{p,j+1}^n x^j D_x^{pn-n+j} S, \text{ where } T_{p,j+1}^{n+1} = ((p-1)n + j + 1)T_{p,j+1}^n + T_{p,j}^n. \quad (6.25)$$

7 Intermediate Operator Techniques

Remark 7.1 Throughout this chapter, we assume n and α are nonnegative integers, while x and z are real or complex numbers. We also use the convention that $D_x^n f(x)$ denotes the n^{th} derivative of f with respect to x .

7.1 Change of Variable Formula

Assume y is a function of z and $x = e^z$. Then,

$$D_x^n y = e^{-nz} D_z (D_z - 1) \dots (D_z - n + 1) y = \frac{n!}{x^n} \binom{D_z}{n} y. \quad (7.1)$$

7.2 The Operator $(xD_x)^n$

Grunert's Formula

Let y be function of z . Assume $x = e^z$. Then,

$$(xD_x)^n y = D_z^n y = \sum_{j=0}^n \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^n \binom{D_z}{j} y \quad (7.2)$$

7.3 The Operator $\left(\frac{1}{x}D_x\right)^n$

Let y be a function of z . Assume $x = e^z$. Then,

$$\left(\frac{1}{x}D_x\right)^n y = \frac{n!}{x^{2n}} \sum_{j=0}^n C_j^n 2^{n-j} D_z^j y = \frac{n!}{x^{2n}} \sum_{j=0}^n C_j^n 2^{n-j} (xD_x)^j y, \quad (7.3)$$

where C_j^n is the coefficient of x^j in $\binom{x}{n}$, i.e. $\binom{x}{n} = \sum_{j=0}^n C_j^n x^j$.

7.3.1 Application of Equation (7.3)

Assume C_j^n is the coefficient of x^j the series expansion of $\binom{x}{n}$. Define $a_j(n)$ as

$$a_j(n) = \frac{(-1)^j}{j!} \sum_{k=0}^j (-1)^k \binom{j}{k} k^n.$$

Let S be a function of x . Then,

$$\left(\frac{1}{x}D_x\right)^n S = n! \sum_{k=0}^n \sum_{j=k}^n x^{k-2n} 2^{n-j} C_j^n a_k(j) D_x^k S. \quad (7.4)$$

7.4 The Operator $(x^n D_x^n)^\alpha$

Remark 7.2 A reference for the formulas in this section is "On a Class of Finite Sums" by Leonard Carlitz, *The American Math. Monthly*, Nov. 1930, Vol. 37, No. 9, pp. 472-479.

Assume S is a function of x . Then,

$$(x^n D_x^n)^\alpha S = n!^\alpha \sum_{k=1}^{n(\alpha-1)+1} \frac{x^{k+n-1} D_x^{k+n-1} S}{(k+n-1)!} * \sum_{j=0}^{k+n-1} (-1)^j \binom{k+n-1}{j} \binom{k+n-1-j}{n}^\alpha, \quad \alpha \geq 1. \quad (7.5)$$

7.4.1 Specific Evaluations of Equation (7.5)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^\alpha = \frac{1}{x^{2n}} \sum_{k=0}^{2n} \frac{(-1)^k}{k!^\alpha} (x^k D_x^k)^\alpha x^{2n} \quad (7.6)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^{\alpha+1} = \frac{1}{x^{2n}} \sum_{k=0}^{2n} \frac{(-1)^k}{k!^\alpha} \binom{2n}{k} (x^k D_x^k)^\alpha x^{2n} \quad (7.7)$$

7.5 The Operator $(D_x^n x^n)^\alpha$

Let S be a function of x . Assume $n\alpha \leq m$. Then,

$$(D_x^n x^n)^\alpha S = n!^\alpha \sum_{k=0}^m \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n}^\alpha \frac{x^k}{k!} D_x^k S. \quad (7.8)$$

7.6 The Operator $D_x^{D_x}$

$$(D_x^{D_x}) f(x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} D_z^n \sum_{k=0}^n (-1)^{n-k} D_x^k f(x), \quad \text{where } z = e^x \quad (7.9)$$

$$(D_x^{D_x}) f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^n C_j^n \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} D_x^{k+j} f(x), \quad (7.10)$$

where C_j^n is the coefficient of x^j the series expansion of $\binom{x}{n}$.

7.7 Derivatives of a Class of Transcendental Functions

Remark 7.3 For this section, we define $E_0(x) = x$, $E_1(x) = e^x$, $E_2(x) = e^{e^x}$, and $E_n(x) = e^{E_{n-1}(x)}$. We define $a_j(n)$ as namely,

$$a_j(n) = \frac{(-1)^j}{j!} \sum_{k=0}^j (-1)^k \binom{j}{k} k^n.$$

Finally, for this section only, we assume r is a nonnegative integer.

$$D_x^n E_r(x) = E_r(x) \sum_{i=1}^{n-1} \sum_{k_i=0}^{k_{i+1}} a_{k_i}(k_{i+1}) e^{k_i E_{r-1-i}(x)}, \quad n \geq 2, \quad k_n = n \quad (7.11)$$

7.7.1 Specific Examples of Equation (7.11)

$$D_x^n e^{e^x} = e^{e^x} \sum_{j=0}^n a_j(n) e^{jx} \quad (7.12)$$

$$D_x^n e^{e^{e^x}} = e^{e^{e^x}} \sum_{k=0}^j a_k(j) e^{kx} \sum_{\alpha=0}^k a_\alpha(k) e^{\alpha e^x} \quad (7.13)$$

$$D_x^n e^{e^{e^{e^x}}} = e^{e^{e^{e^x}}} \sum_{j=0}^n a_j(n) e^{jx} \sum_{\beta=0}^j a_\beta(j) e^{\beta e^x} \sum_{\alpha=0}^{\beta} a_\alpha(\beta) e^{\alpha e^{e^x}} \quad (7.14)$$

7.8 Derivatives of x^x

Remark 7.4 In this section, we define the sequence $\{F_x(k)\}_{k=0}^\infty$ by $F_0(x) = 1$, $F_1(x) = 1$, and $F_k(x) = -D_x F_{k-1}(x) + \frac{k-1}{x} F_{k-2}(x)$, whenever $k \geq 2$. We assume x is nonzero.

$$D_x^n (x^x) = x^x \sum_{k=0}^n (-1)^k \binom{n}{k} (1 + \ln x)^{n-k} F_k(x) \quad (7.15)$$

8 Advanced Operator Techniques

Remark 8.1 Throughout this chapter, we assume n and r are nonnegative integers, while x is a nonzero real or complex number. We assume, unless otherwise specified, that y is a function of x . We will use the convention that $D_x^n y \equiv D^n y$ is n^{th} derivative of y with respect to x .

8.1 The Non-Linear Operator $(x + D)^n$

8.1.1 Definition of $(x + D)^n$

$$(x + D)^n y \equiv \underline{D}_1^n y = \prod_{k=1}^n (x + D) y, \quad D_1^0 \equiv 1 \quad (8.1)$$

8.1.2 An Expansion of \underline{D}_1^n

$$\underline{D}_1^n y = \sum_{j=0}^n \binom{n}{j} [(x + D)^{j-1} x] D^{n-j} y, \quad (8.2)$$

8.1.3 Alternative Expansion for \underline{D}_1^n

$$\underline{D}_1^n y = \sum_{j=0}^n \binom{n}{j} A_j(x) D^{n-j} y, \quad (8.3)$$

where the $A_j(x)$ are polynomials of degree j in x such that

$$A_{n+1}(x) = xA_n(x) + nA_{n-1}(x), \quad A_0(x) = 1. \quad (8.4)$$

8.1.4 Properties of $A_n(x)$

$$D_x^r A_n(x) = r! \binom{n}{r} A_{n-r}(x) \quad (8.5)$$

$$A_n(x) = n \int_0^x A_{n-1}(z) dz + A_n(0) \quad (8.6)$$

$$D_x^n e^{x^2} = (\sqrt{2})^n e^{x^2} A_n(\sqrt{2}x) \quad (8.7)$$

8.1.5 Explicit Formulas for $A_n(x)$

$$A_n(x) = \sum_{k=0}^n \binom{n}{k} x^k A_{n-k}(0) \quad (8.8)$$

Remark 8.2 For real x , let $[x]$ denote the floor of x .

$$A_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \binom{n-k}{k} \frac{k!}{2^k} x^{n-2k} \quad (8.9)$$

$$A_n(x) = \frac{1}{(i\sqrt{2})^n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{n-k}{k} k! (i\sqrt{2}x)^{n-2k}, \quad i = \sqrt{-1} \quad (8.10)$$

8.1.6 Properties of $(x + D)^n$

Order Property

$$(x + D)^{n+1}y = (x + D)(x + D)^ny = (x + D)^n(x + D)y \quad (8.11)$$

Product Property: Assume u and v are functions of x . Then,

$$(x + D)^n uv = \sum_{k=0}^n \binom{n}{k} D^k v (x + D)^{n-k} u. \quad (8.12)$$

8.2 The Non-Linear Operator $(x - D)^n$

8.2.1 Definition of $(x - D)^n$

$$(x - D)^n y \equiv \underline{D}_2^n y = \prod_{k=1}^n (x - D)y, \quad D_2^0 \equiv 1 \quad (8.13)$$

8.2.2 An Expansion of \underline{D}_2^n

$$\underline{D}_2^n y = \sum_{j=0}^n (-1)^j \binom{n}{j} \left[\frac{1}{a} (x - D)^{n-k} a \right] D^j y, \quad a \text{ any nonzero constant} \quad (8.14)$$

8.2.3 Alternative Expansion for \underline{D}_2^n

$$\underline{D}_2^n y = \sum_{j=0}^n (-1)^j \binom{n}{j} \bar{A}_{n-j}(x) D^j y, \quad (8.15)$$

where the $\bar{A}_j(x)$ are polynomials of degree j in x such that

$$\bar{A}_{n+1}(x) = x\bar{A}_n(x) - n\bar{A}_{n-1}(x), \quad \bar{A}_0(x) = 1. \quad (8.16)$$

8.2.4 Properties of $\bar{A}_n(x)$

$$D_x^r \bar{A}_n(x) = r! \binom{n}{r} \bar{A}_{n-r}(x) \quad (8.17)$$

$$\bar{A}_n(x) = n \int_0^x \bar{A}_{n-1}(z) dz + \bar{A}_n(0) \quad (8.18)$$

8.2.5 Explicit Formulas for $\bar{A}_n(x)$

$$\bar{A}_n(x) = \sum_{k=0}^n \binom{n}{k} x^k \bar{A}_{n-k}(0) \quad (8.19)$$

$$\bar{A}_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{n-k}{k} \frac{k!}{2^k} x^{n-2k} \quad (8.20)$$

$$\bar{A}_n(x) = \frac{1}{(\sqrt{2})^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{n-k}{k} k! (\sqrt{2}x)^{n-2k} \quad (8.21)$$

$$\bar{A}_n(x) = (-1)^n e^{\frac{x^2}{2}} D_x^n e^{-\frac{x^2}{2}} \quad (8.22)$$

8.2.6 Properties of $(x - D)^n$

Order Property

$$(x - D)^{n+1}y = (x - D)(x - D)^n y = (x - D)^n (x - D)y \quad (8.23)$$

Product Property: Assume u and v are functions of x . Then,

$$(x - D)^n uv = \sum_{k=0}^n \binom{n}{k} D^k v (x - D)^{n-k} u. \quad (8.24)$$

8.3 Advanced Properties of $A_n(x)$ and $\bar{A}_n(x)$

8.3.1 Addition Properties

$$A_n(x + y) = \sum_{k=0}^n \binom{n}{k} x^{n-k} A_k(y) \quad (8.25)$$

$$\bar{A}_n(x + y) = \sum_{k=0}^n \binom{n}{k} x^{n-k} \bar{A}_k(y) \quad (8.26)$$

8.3.2 In the Expansion of x^n

$$x^n = \sum_{k=0}^n \binom{n}{k} A_{n-k}(x) \bar{A}_k(0) \quad (8.27)$$

$$x^n = \sum_{k=0}^n \binom{n}{k} \bar{A}_{n-k}(x) A_k(0) \quad (8.28)$$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} \bar{A}_{n-k}(x) A_k(y) \quad (8.29)$$

$$(x-y)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{A}_{n-k}(x) A_k(y) \quad (8.30)$$

8.3.3 Operator Evaluations

$$A_n(x-D)y = \sum_{k=0}^n (-1)^k \binom{n}{k} x^{n-k} D^k y \quad (8.31)$$

$$\bar{A}_n(x+D)y = \sum_{k=0}^n \binom{n}{k} x^{n-k} D^k y \quad (8.32)$$

8.3.4 Convolution Formulas

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} A_k(x) A_{2n-k}(x) = \frac{(2n)!}{n!} \quad (8.33)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \bar{A}_k(x) \bar{A}_{2n-k}(x) = (-1)^n \frac{(2n)!}{n!} \quad (8.34)$$

8.4 The Linear Operator $(D + a)^n$

Remark 8.3 Throughout this section, we assume a and b are nonzero real or complex numbers. We will also assume u and v are functions of x .

8.4.1 Definition of $(D + a)^n$

$$(D + a)^n u = \sum_{k=0}^n \binom{n}{k} a^{n-k} D^k u \quad (8.35)$$

8.4.2 Product Rule

$$(D + a)^n uv = \sum_{k=0}^n \binom{n}{k} D^k v (D + a)^{n-k} u \quad (8.36)$$

$$(D + a - b)^n uv = \sum_{k=0}^n \binom{n}{k} (D + a)^k u (D - b)^{n-k} v \quad (8.37)$$

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} (D + a)^k u (D - a)^{n-k} v \quad (8.38)$$

8.4.3 Polynomial Operator $f(D + a)$

Remark 8.4 Throughout this section, we assume $f(x)$ is a polynomial of degree n , namely, $f(x) = \sum_{k=0}^n a_k x^k$. We will also use the notation $f^{(k)}(x)$ to denote $D_x^k f(x)$.

$$f(D + a - b)uv = \sum_{k=0}^n \frac{(D + a)^k u}{k!} f^{(k)}(D - b)v \quad (8.39)$$

$$f(D)uv = \sum_{k=0}^n \frac{(D + a)^k u}{k!} f^{(k)}(D - a)v \quad (8.40)$$

Exponential Shift Theorem

$$f(D - a + b)e^{ax}v = e^{ax}f(D + b)v \quad (8.41)$$

Applications of Equation (8.41)

$$f(D)(e^{ax}v) = e^{ax}f(D+a)v \quad (8.42)$$

$$D^n(e^xv) = e^x(D+1)^n v = e^x \sum_{k=0}^n \binom{n}{k} D^k v \quad (8.43)$$

Remark 8.5 An alternative derivation for the exponential shift theorem and generalized Leibniz formula are given by C. J. Coe, “The Generalized Leibniz Formula”, *The American Math. Monthly*, Sept. 1950, Vol. 57, No. 7, pp. 459-466.

9 Shifted Legendre Polynomials $\tilde{P}_n(x)$

Remark 9.1 In this chapter, we assume n and r are nonnegative integers, while x and a are real or complex numbers. If x is a real number, we use $[x]$ to denote the floor of x .

9.1 Definition of $\tilde{P}_n(x)$

$$\tilde{P}_n(x) = \sum_{k=0}^n \binom{n}{k}^2 x^{n-k} = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (x-1)^k \quad (9.1)$$

9.2 Expansion of $\tilde{P}_n(x)$

$$\tilde{P}_n(x) = \sum_{j=0}^n (-1)^j x^j \sum_{k=j}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{k}{j} \quad (9.2)$$

9.2.1 Applications of Equation (9.2)

$$\sum_{k=j}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{k}{j} = (-1)^j \binom{n}{j}^2 \quad (9.3)$$

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{2n-k}{n} k = n^2, \quad n \geq 1 \quad (9.4)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{n+k}{n} \quad (9.5)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{2n-k} \binom{2k}{k} \binom{4n-k}{4n} \quad (9.6)$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} = 0, \quad \text{where } p \text{ is an integer} \quad (9.7)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 &= (-1)^n \binom{2n}{n}^2 \\ &+ \sum_{k=0}^n \binom{n+k-1}{n} \sum_{j=0}^k (-1)^j \binom{2n+1}{j} \binom{3n-j+k}{n} \binom{n-1-j+k}{n}, \quad n \geq 1 \end{aligned} \quad (9.8)$$

$$\sum_{k=0}^{2n+1} \binom{2n+k}{2n+1} \sum_{j=0}^{k-1} (-1)^j \binom{4n+3}{j} \binom{6n+3-j+k}{2n+1} \binom{2n-j+k}{2n+1} = \binom{4n+2}{2n+1}^2 \quad (9.9)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 \frac{1}{\binom{rn+n}{k}} = \frac{\binom{rn}{n}^2}{\binom{rn+n}{2n} \binom{2n}{n}} \quad (9.10)$$

$$\sum_{j=0}^n \binom{n}{j}^2 f(j) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \sum_{j=0}^k (-1)^j \binom{k}{j} f(j) \quad (9.11)$$

Examples of Equation (9.11)

$$\sum_{j=0}^n \binom{n}{j}^2 j^r = \sum_{k=0}^r \binom{n}{k} \binom{2n-k}{n} \sum_{j=0}^k (-1)^{j+k} \binom{k}{j} j^r \quad (9.12)$$

$$\sum_{j=0}^n \binom{n}{j}^2 \frac{1}{2j+1} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \frac{1}{\binom{k+\frac{1}{2}}{k}} \quad (9.13)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{-n-1}{n-k} \frac{1}{\binom{k+r}{k}} = r(-1)^n \sum_{j=0}^n \binom{n}{j}^2 \frac{1}{j+r}, \quad r \geq 1 \quad (9.14)$$

$$\sum_{j=0}^n \binom{n}{j}^2 \binom{j}{n-k} = \binom{n}{k} \binom{n+k}{k} = \binom{n}{k} \binom{2n-k}{n}, \quad 0 \leq k \leq n \quad (9.15)$$

Remark 9.2 The following identity, due to E. T. Bell, is Problem 3457 of the *American Math. Monthly*, 1930, pp. 507-508.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} 2^{2n-2k} = \frac{1+(-1)^n}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor}^2 \quad (9.16)$$

9.3 Expansion of $\tilde{P}_n^2(x)$

$$\tilde{P}_n^2(x) = \sum_{k=0}^{2n} x^k \sum_{j=\lfloor \frac{k}{n+1} \rfloor (k-n)}^{k-\lfloor \frac{k+1}{n+1} \rfloor (k-n)} \binom{n}{j}^2 \binom{n}{k-j}^2 \quad (9.17)$$

9.3.1 Applications of Equation (9.17)

$$\sum_{k=0}^n \binom{n}{k}^4 = \sum_{j=0}^n \binom{n}{j} \binom{2n-j}{n} \sum_{k=0}^j (-1)^k \binom{k+n}{n} \binom{n}{j-k} \binom{n-k+j}{n} \quad (9.18)$$

$$\begin{aligned} \sum_{k=0}^n \left[\binom{n}{k}^4 - (-1)^k \binom{n}{k}^3 \right] &= \sum_{j=0}^{n-1} \binom{n}{j} \binom{2n-j}{n} \\ &\quad * \sum_{k=0}^j (-1)^k \binom{k+n}{n} \binom{n}{j-k} \binom{n-k+j}{n}, \quad n \geq 1 \end{aligned} \quad (9.19)$$

$$\sum_{k=0}^n \binom{n}{k}^4 = \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{2n-j}{n} \sum_{k=0}^j (-1)^k \binom{n}{k}^2 \binom{j}{k} \quad (9.20)$$

9.4 Three Expansions of $(x - 1)^{2n} \tilde{P}_{2n}(x)$

9.4.1 First Expansion of $(x - 1)^{2n} \tilde{P}_{2n}(x)$

$$(x - 1)^{2n} \tilde{P}_{2n}(x) = \sum_{k=0}^{4n} x^{4n-k} \sum_{j=\lfloor \frac{k}{2n+1} \rfloor (k-2n)}^{k-\lfloor \frac{k+1}{2n+1} \rfloor (k-2n)} (-1)^{k-j} \binom{2n}{j}^2 \binom{2n}{k-j} \quad (9.21)$$

9.4.2 Second Expansion of $(x - 1)^{2n} \tilde{P}_{2n}(x)$

$$(x - 1)^{2n} \tilde{P}_{2n}(x) = \sum_{j=0}^m (-1)^{j+2n} x^j \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n-k}{2n} \binom{2n+k}{j}, \quad m \geq 4n \quad (9.22)$$

Applications of Equations (9.21) and (9.22)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{2n-k} \binom{2k}{k} \binom{4n-k}{2n} \quad (9.23)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n} \binom{3n}{2n} \quad (9.24)$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!^3} \sum_{j=0}^{\infty} \frac{x^j}{j!^3} = \sum_{k=0}^{\infty} (-1)^k \frac{(3k)!}{(2k)!^3 k!^3} x^{2k} \quad (9.25)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k}^2 \binom{2n}{n-k} = \sum_{k=0}^n (-1)^{k+n} \binom{2n}{k+n} \binom{3n-k}{2n} \binom{3n+k}{3n} \quad (9.26)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k}^2 \binom{2n}{n-k} = \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{2n+k}{2n} \binom{4n-k}{3n} \quad (9.27)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k}^2 \binom{2n}{n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n-k}{2n} \binom{2n+k}{n} \quad (9.28)$$

9.4.3 Third Expansion of $(x - 1)^{2n} \tilde{P}_{2n}(x)$

$$(x - 1)^{2n} \tilde{P}_{2n}(x) = \sum_{i=0}^{2n} \binom{2n}{i}^2 \sum_{j=0}^i \binom{i}{j} \sum_{k=0}^{2n+j} (-1)^{2n+j-k} \binom{2n+j}{k} x^k \quad (9.29)$$

Application of Equation (9.29)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \sum_{k=0}^{2n} \binom{2n}{k}^2 \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{2n+j}{2n} \quad (9.30)$$

9.5 Expansion of $(a + x)^{2n} \tilde{P}_{2n}(x)$

$$(a + x)^{2n} \tilde{P}_{2n}(x) = \sum_{k=0}^{4n} x^{4n-k} \sum_{j=\lceil \frac{k}{2n+1} \rceil (k-2n)}^{k - \lceil \frac{k+1}{2n+1} \rceil (k-2n)} \binom{2n}{j}^2 \binom{2n}{k-j} a^{k-j}, \quad a \neq 0 \quad (9.31)$$

9.6 Expansion of $\tilde{P}_{2n}^2(x)$

$$\tilde{P}_{2n}^2(x) = \sum_{k=0}^{4n} x^k \sum_{j=\lceil \frac{k}{2n+1} \rceil (k-2n)}^{k - \lceil \frac{k+1}{2n+1} \rceil (k-2n)} \binom{2n}{j}^2 \binom{2n}{k-j}^2 \quad (9.32)$$

9.6.1 Application of Equation (9.32)

$$\begin{aligned} \sum_{j=0}^n \binom{2n}{j}^3 \binom{2n}{j+n}^2 &= \sum_{j=0}^n \binom{2n}{j+n} \binom{3n-j}{2n} \\ &\quad * \sum_{k=0}^j (-1)^k \binom{3n+k}{3n} \binom{2n}{j-k} \binom{2n-k+j}{2n} \end{aligned} \quad (9.33)$$

9.7 Expansions of $(x - 1)^n \tilde{P}_n(x)$

$$(x - 1)^n \tilde{P}_n(x) = \sum_{k=0}^{2n} (-1)^k x^k \sum_{j=0}^{2n-k} (-1)^j \binom{n}{j}^2 \binom{n}{2n-k-j} \quad (9.34)$$

$$(x - 1)^n \tilde{P}_n(x) = \sum_{k=0}^{2n} (-1)^k x^k \sum_{j=0}^n (-1)^{n+j} \binom{n}{j} \binom{2n-j}{n} \binom{n+j}{k} \quad (9.35)$$

9.7.1 Application of Equations (9.34) and (9.35)

$$\sum_{j=0}^{2n-k} (-1)^j \binom{n}{j}^2 \binom{n}{2n-k-j} = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} \binom{2n-j}{n} \binom{n+j}{k} \quad (9.36)$$

$$\sum_{j=0}^{2n-k} (-1)^j \binom{n}{j}^2 \binom{n}{2n-k-j} = \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+j}{n} \binom{2n-j}{k} \quad (9.37)$$

10 Legendre Polynomials $P_n(x)$

Remark 10.1 In this chapter, we assume n and r are nonnegative integers, while x , y , and t are real or complex numbers. If x is a real number, we use $[x]$ to denote the floor of x . We also use the convention that $D_x^n f(x)$ is the n^{th} derivative of $f(x)$ with respect to x .

10.1 Three Ways to Define $P_n(x)$

10.1.1 Generating Function for $P_n(x)$

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1 - 2xt + t^2)^{-\frac{1}{2}} \quad (10.1)$$

10.1.2 Rodrigues Formula

$$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2 - 1)^n \quad (10.2)$$

10.1.3 Explicit Summation for $P_n(x)$

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{x^{n-2k}}{2^n} \quad (10.3)$$

10.1.4 Two Alternative Summations for $P_n(x)$

$$P_n(x) = \left(\frac{x-1}{2} \right)^n \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{x+1}{x-1} \right)^k \quad (10.4)$$

$$P_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} \left(\frac{x-1}{2}\right)^{n-k} \quad (10.5)$$

10.2 Binomial Identities from the Basic Definitions of $P_n(x)$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n \quad (10.6)$$

$$\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k-2}{n} = 2^n - n - 1, \quad n \geq 2 \quad (10.7)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{1}{2^{2k}} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k}^2 3^k = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} 2^k \quad (10.8)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} = \sum_{k=0}^n \binom{n}{k}^2 2^k = \left(\frac{3}{2}\right)^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{1}{9^k} \quad (10.9)$$

$$\begin{aligned} \left(\frac{5}{2}\right)^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{1}{5^{2k}} &= 2^n \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{3}{2}\right)^k \\ &= 2^n \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} \frac{1}{2^k} \end{aligned} \quad (10.10)$$

Remark 10.2 The following two identities, due to L. Carlitz, are the solutions to Problem 352 of *The Math. Magazine*, Vol. 32, No. 1, Sept.-Oct. 1958, pp. 47-48. We use the convention that $((x^n))f(x)$ is the coefficient of x^n in the power series expansion of $f(x)$.

$$((x^n))(1-x^2)^n P_n\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^n \binom{n}{k}^3 \quad (10.11)$$

$$((x^n))(1-x)^{2n} \left(P_n\left(\frac{1+x}{1-x}\right)\right)^2 = \sum_{k=0}^n \binom{n}{k}^4 \quad (10.12)$$

10.3 Connections to Shifted Legendre Polynomials $\tilde{P}_n(x)$

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \tilde{P}_n\left(\frac{x+1}{x-1}\right), \quad x \neq 1 \quad (10.13)$$

$$\tilde{P}_n(t) = (t-1)^n P_n\left(\frac{t+1}{t-1}\right), \quad t \neq 1 \quad (10.14)$$

10.3.1 Expansions of $\tilde{P}_n(x)$ Using Equation (10.14)

$$(x-1)^n \tilde{P}_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} (x+1)^{n-2k} (x-1)^{n+2k} \quad (10.15)$$

$$(x-1)^{2n} \tilde{P}_{2n}(x) = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{4n-2k}{2n} (x+1)^{2n-2k} (x-1)^{2n+2k} \quad (10.16)$$

Applications of Equations (10.15) and (10.16)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{4n-2k}{2n} \sum_{j=2k}^{2n} (-1)^j \binom{2n+2k}{j} \binom{2n-2k}{2n-j} \quad (10.17)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(6n)!}{2^{2n} (2n)!^3} \sum_{k=0}^n \binom{2n}{k} \binom{2n}{n-k} \frac{1}{\binom{6n}{4n-2k}} \quad (10.18)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \frac{(-1)^n}{2^{2n} \binom{2n}{n}} \sum_{k=0}^n \binom{n+k}{n} \binom{2n+2k}{n+k} \binom{4n-2k}{2n-k} \binom{2n-k}{n} \quad (10.19)$$

10.4 Laplace Integral for $P_n(x)$ and Various Applications

10.4.1 Laplace Integral for $P_n(x)$

$$P_n(x) = \int_0^1 (x + \sqrt{x^2-1} \cos 2\pi t)^n dt \quad (10.20)$$

10.4.2 Good's Formula for $P_n(x)$

Remark 10.3 The following formula is found in “A new finite series for Legendre polynomials”, by I. J. Good, *Proc. Cambridge Philosophical Society*, Vol. 51, 1955, pp. 385-388.

Good's Formula

$$P_n(x) = \frac{1}{r} \sum_{k=0}^{r-1} \left(x + \sqrt{x^2 - 1} \cos \frac{2\pi k}{r} \right), \text{ for integral } r > n, r \neq 0 \quad (10.21)$$

Identities from the Proof of Good's Formula

$$P_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \frac{(x^2 - 1)^{\frac{k}{2}} (x - (x^2 - 1)^{\frac{1}{2}})^{n-k}}{2^k} \quad (10.22)$$

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \binom{2k}{k} \frac{x^{n-2k} (x^2 - 1)^k}{2^{2k}} \quad (10.23)$$

$$\sum_{k=j}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{k-j} \binom{n-k}{k} 2^{n-2k} = \binom{2n-2j}{n} \quad (10.24)$$

10.4.3 A Convolution Identity

Remark 10.4 The following identity is from “Inverse Elliptic Functions and Legendre Polynomials” by R. P. Kelisky in *The American Math. Monthly*, Vol. 66, 1959, pp. 480-485.

$$2^{2n} x^n P_n \left(\frac{x + x^{-1}}{2} \right) = \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k} = \frac{2^{2n+1}}{\pi} \int_0^{\frac{\pi}{2}} (x^2 \sin^2 t + \cos^2 t)^n dt \quad (10.25)$$

Applications of Equation (10.25)

$$(-1)^n x^n P_n \left(\frac{x + x^{-1}}{2} \right) = \sum_{k=0}^n \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{n-k} x^{2k} \quad (10.26)$$

$$(-1)^n x^n P_n \left(\frac{x + x^{-1}}{2} \right) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k - \frac{1}{2}}{n} x^{2k} \quad (10.27)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{2k}{k+j} \frac{1}{2^{2k}} = \frac{(-1)^n + (-1)^j}{2^{2n+1}} \binom{n+j}{\frac{n+j}{2}} \binom{n-j}{\frac{n-j}{2}}, \quad (10.28)$$

j integral, $0 \leq j \leq n$

10.5 Derivatives of $P_n(x)$

$$D_x^r P_n(x) = \frac{r!}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{r} x^{n-2k-r} \quad (10.29)$$

$$D_x^r P_n(x)|_{x=0} = r! \frac{(-1)^{\frac{n-r}{2}}}{2^n} \binom{n}{\frac{n-r}{2}} \binom{n+r}{n} \frac{1+(-1)^{n-r}}{2} \quad (10.30)$$

10.5.1 Applications of Equation (10.30)

$$\sum_{k=j}^n (-1)^k \binom{n}{k} \binom{2k}{k-j} \frac{1}{2^k} = \frac{(-1)^j}{2^n} \binom{n}{\frac{n-j}{2}} \frac{1+(-1)^{n-j}}{2}, \quad 0 \leq j \leq n \quad (10.31)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{2^k} = \frac{1}{2^n} \binom{n}{\frac{n}{2}} \frac{1+(-1)^n}{2} = \frac{1}{2^{2n}} \binom{2n}{n} \quad (10.32)$$

10.5.2 Grosswald's Formula

Remark 10.5 The identities of this section are found in Emil Grosswald's "On Sums Involving Binomial Coefficients", *American Math. Monthly*, Vol. 60, No. 3, March 1953, pp. 179-181.

Grosswald's Formula

$$D_x^r P_n(x)|_{x=0} = \frac{r!}{2^r} \binom{n}{r} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+k+r}{n} \frac{1}{2^k} \quad (10.33)$$

Applications of Equation (10.33)

$$\sum_{k=0}^{n-r} (-1)^k \binom{n}{k+r} \binom{n+k+r}{k} \frac{1}{2^k} = (-1)^{\frac{n-r}{2}} 2^{r-n} \binom{n}{\frac{n+r}{2}} \frac{1+(-1)^{n-r}}{2} \quad (10.34)$$

$$\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+k+r}{n} \frac{1}{2^k} = (-1)^{\frac{n-r}{2}} \frac{2^{r-n}}{\binom{n}{r}} \binom{n}{\frac{n-r}{2}} \binom{n+r}{n} \frac{1+(-1)^{n-r}}{2} \quad (10.35)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k+2r}{n} \frac{1}{2^k} = (-1)^{\frac{n}{2}} \frac{2^{-n}}{\binom{n+r}{r}} \binom{n+r}{\frac{n}{2}} \binom{n+2r}{r} \frac{1+(-1)^n}{2} \quad (10.36)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n+k}{3n} \frac{1}{2^k} = (-1)^n \binom{4n}{n} \frac{1}{2^{2n}} \quad (10.37)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{6n+k}{4n} \frac{1}{2^k} = (-1)^n \binom{4n}{n} \binom{6n}{2n} \frac{1}{2^{2n} \binom{4n}{2n}} \quad (10.38)$$

$$\sum_{k=0}^{3n} (-1)^k \binom{3n}{k} \binom{5n+k}{4n} \frac{1}{2^k} = (-1)^{\frac{3n}{2}} \frac{1}{2^{3n} \binom{4n}{n}} \binom{4n}{\frac{3n}{2}} \binom{5n}{n} \frac{1 + (-1)^n}{2} \quad (10.39)$$

Extensions of Equation (10.34)

Remark 10.6 *The following two identities are from T. S. Nanjundiah's "On a Formula of Grosswald", American Math. Monthly, Vol. 61, No. 10, December 1954, pp. 700-702.*

$$\sum_{k=0}^n (-2)^k \binom{\mu}{k} \binom{2\mu-2k}{n-k} = \frac{(-1)^n + 1}{2} \binom{\mu}{\frac{n}{2}}, \quad u \text{ a real number} \quad (10.40)$$

$$\sum_{k=0}^n (-2)^k \binom{\mu}{k} \binom{2\mu-k}{n-k} = (-1)^{\frac{n}{2}} \frac{(-1)^n + 1}{2} \binom{\mu}{\frac{n}{2}}, \quad u \text{ a real number} \quad (10.41)$$

10.5.3 Double Summation Formula for $P_n(x)$

$$P_n(x) = \sum_{r=0}^n \binom{n}{r} \frac{x^r}{2^r} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+k+r}{n} \frac{1}{2^k} \quad (10.42)$$

10.5.4 Extensions of Equation (10.29)

$$D_x^r(x) P_n(x)|_{x=\pm 1} = (\pm 1)^{n+r} \binom{n}{r} \binom{n+r}{r} \frac{r!}{2^r} \quad (10.43)$$

Identities from the Proof of Equation (10.43)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n+r} = 2^{n-r} \binom{n}{r} \quad (10.44)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{r} = 2^{n-r} \binom{n}{r} \binom{n+r}{r} \quad (10.45)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{4n-2k}{2n} \binom{2n-2k}{n} = 2^n \binom{2n}{n} \binom{3n}{2n} \quad (10.46)$$

Application of Equation (10.44)

$$\sum_{r=0}^n \binom{n}{r}^2 \frac{1}{2^r} = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{3n-2k}{2n} \quad (10.47)$$

Generalization of Equation (10.44)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{x}{k} \binom{2x-2k}{n-2k} = 2^n \binom{x}{n} \quad (10.48)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+x}{k} \binom{2n+2x-2k}{n-2k} = 2^n \binom{x+n}{n} \quad (10.49)$$

10.6 Differential and Difference Recurrence Relations

10.6.1 Differential Recurrence Relations

$$nP_n(x) - xP'_n + P'_{n-1}(x) = 0, \quad n \geq 1 \quad (10.50)$$

$$nP_{n-1}(x) - P'_n(x) + xP'_{n-1}(x) = 0, \quad n \geq 1 \quad (10.51)$$

$$(2n+1)P_n(x) - P'_{n+1}(x) + P'_{n-1}(x) = 0, \quad n \geq 1 \quad (10.52)$$

$$nP_{n-1}(x) + (x^2-1)P'_n(x) - nxP_n(x) = 0, \quad n \geq 1 \quad (10.53)$$

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0 \quad (10.54)$$

$$P'_n(x) = \frac{n}{1-x^2} (P_{n-1}(x) - xP_n(x)), \quad n \geq 1 \quad (10.55)$$

10.6.2 Difference Recurrence Relation

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad n \geq 1 \quad (10.56)$$

10.6.3 Christoffel's Identity

Remark 10.7 A reference for Christoffel's Identity is *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, 1941. The reader is also referred to E. B. Christoffel's, "Über die Gausssche Quadratur und eine Verallgemeinerung derselben", *Journal für die reine und angewandte Mathematik*, Vol. 55, 1858, pp. 61-82.

Christoffel's Identity

$$(x-y) \sum_{k=0}^n (2k+1)P_k(x)P_k(y) = (n+1)(P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)) \quad (10.57)$$

10.7 Evaluation of $\int_{-1}^1 P_n(x)P_r(x) dx$

$$\int_{-1}^1 P_n(x)P_r(x) dx = \begin{cases} 0, & n \neq r \\ \frac{2}{2n+1}, & n = r \end{cases} \quad (10.58)$$

$$\int_{-1}^1 P_n(x)P_r(x) dx = \frac{2(-1)^{n+r}}{n+r+1} \sum_{k=0}^{n+r} \frac{(-1)^k}{\binom{n+r}{k}} \sum_{j=0}^k \binom{n}{j}^2 \binom{r}{k-j}^2 \quad (10.59)$$

10.7.1 Formulas Associated with Equation (10.59)

$$\sum_{k=0}^{n+r} (-1)^k \sum_{j=0}^k \binom{n}{j}^2 \binom{r}{k-j}^2 = (-1)^{[\frac{n}{2}] + [\frac{r}{2}]} \binom{n}{[\frac{n}{2}]} \binom{r}{[\frac{r}{2}]} \frac{1 + (-1)^n}{2} \frac{1 + (-1)^r}{2} \quad (10.60)$$

$$\sum_{k=0}^{2n+2r} (-1)^k \sum_{j=0}^k \binom{2n}{j}^2 \binom{2r}{k-j}^2 = (-1)^{n+r} \binom{2n}{n} \binom{2r}{r} \quad (10.61)$$

10.8 Evaluation of $\int_{-1}^1 x^r P_n(x) dx$

$$\int_{-1}^1 x^r P_n(x) dx = \begin{cases} \frac{1}{2^n} \frac{\binom{r}{n}}{\binom{n+r+1}{n}} \frac{1+(-1)^{r+n}}{r-n-1}, & r \geq n \geq 0 \\ 0, & n > r \geq 0 \end{cases} \quad (10.62)$$

$$\int_{-1}^1 P_n(x) dx = \begin{cases} 0, & n \geq 1 \\ 2, & n = 0 \end{cases} \quad (10.63)$$

$$\int_{-1}^1 x^n P_n(x) dx = \frac{2^{n+1}}{\binom{2n}{n}(2n+1)} \quad (10.64)$$

10.8.1 An Indefinite Integral

$$\int P_n(x) dx = \begin{cases} \frac{xP_n(x) - P_{n-1}(x)}{n+1}, & n \geq 1 \\ x, & n = 0 \end{cases} \quad (10.65)$$

10.9 Polynomial Expansions via Legendre Polynomials

If $f(x)$ is any polynomial of degree n in x ,

$$f(x) = \sum_{k=0}^n P_k(x) \frac{2k+1}{2} \int_{-1}^1 P_k(x) f(x) dx \quad (10.66)$$

10.9.1 Examples of Equation (10.66)

$$x^n = \sum_{k=0}^n \binom{n}{k} \frac{(2k+1)(1+(-1)^{n+k})}{2^{k+1}(n-k+1)\binom{n+k+1}{k}} P_k(x) \quad (10.67)$$

$$x^{2n} = \sum_{k=0}^n \frac{(4k+1)\binom{2n}{n+k}}{(2n+2k+1)\binom{2n+2k}{n+k}} 2^{2k} P_{2k}(x) \quad (10.68)$$

$$\sum_{k=0}^n \frac{(4k+1)\binom{2n}{n+k}}{(2n+2k+1)\binom{2n+2k}{n+k}} 2^{2k} = 1 \quad (10.69)$$

$$x^{2n+1} = \sum_{k=0}^n \frac{\binom{2n}{n+k}}{\binom{2n+2k}{n+k}} \frac{(2n+1)(4k+3)}{(2n+2k+1)(2n+2k+3)} 2^{2k} P_{2k+1}(x) \quad (10.70)$$

$$x^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n-4k+1) \binom{n-k}{k}}{(2n-2k+1) \binom{2n-2k}{n-2k}} 2^{n-2k} P_{n-2k}(x) \quad (10.71)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n-4k+1) \binom{n-k}{k}}{(2n-2k+1) \binom{2n-2k}{n-2k}} 2^{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n-4k+1) \binom{n}{k}}{(2n-2k+1) \binom{2n-2k}{n-2k}} 2^{n-2k} = 1 \quad (10.72)$$

Remark 10.8 An equivalent form of Equation (10.72), namely,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{2n-4k+1} (2n-4k+1) \frac{(n-k+\frac{1}{2})!(n-k)!^2}{k!(2n-2k+1)!^2} = \frac{\sqrt{\pi}}{2^n n!},$$

is found in N. Arley, “Om summation av en række”, *Mat. Tidsskr. B. København*, 1937, pp. 42-44.

Remark 10.9 The following identity, due to René Lagrange, [*Acta Mathematica*, Vol. 52, 1929], is a generalization of Equation (10.72).

$$\sum_{k=0}^n \frac{(x+y+1-2k) \binom{x}{k} \binom{y-k}{n-k}}{(x+y+1-k) \binom{x+y-k}{n}} = 1 \quad (10.73)$$

Applications of Equation (10.73)

$$\sum_{k=0}^n \frac{(2n-4k+1) \binom{n}{k}}{(2n-2k+1) \binom{2n-2k}{n-k}} 2^{2n-2k} = 1 \quad (10.74)$$

$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^n \frac{(2n-4k+1) \binom{n}{k}}{(2n-2k+1) \binom{2n-2k}{n-k}} 2^{2n-2k} = 1 - 2^n \quad (10.75)$$

10.9.2 Application of Equation (10.66) to Integration

If $f(x)$ is any polynomial of degree n in x such that $f(x) = \sum_{k=0}^n a_k P_k(x)$, then

$$\int_{-1}^1 f^2(x) dx = \sum_{k=0}^n \frac{2}{2k+1} a_k^2. \quad (10.76)$$

Applications of Equation (10.76)

$$\sum_{k=0}^n \binom{n}{k}^2 \frac{(2k+1)(1+(-1)^{n+k})}{\left(\frac{n+k+1}{2}\right)^2 2^{2k+1}(n-k+1)^2} = \frac{1}{2n+1} \quad (10.77)$$

$$\sum_{k=0}^n \binom{2n}{2k}^2 \frac{(4k+1)}{\left(\frac{n+k+\frac{1}{2}}{2k}\right)^2 2^{4k}(2n-2k+1)^2} = \frac{1}{4n+1} \quad (10.78)$$

$$\sum_{k=0}^n \frac{\binom{2n}{n+k}^2}{\binom{2n+2k}{n+k}^2} \frac{(4k+1)2^{4k}}{(2n+2k+1)^2} = \frac{1}{4n+1} \quad (10.79)$$

11 Hermite Polynomials $H_n(x)$

Remark 11.1 In this chapter, we assume n and r are nonnegative integers. We assume x, y, z , and t are real or complex numbers. We use the convention that $D_x^r f(x)$ is the r^{th} derivative of $f(x)$. Finally, if x is a real number, we let $[x]$ denote the floor of x .

11.1 Three Definitions for $H_n(x)$

11.1.1 Exponential Generating Function

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \quad (11.1)$$

11.1.2 Operator Definitions

$$H_n\left(\frac{x}{\sqrt{2}}\right) = (-1)^n (\sqrt{2})^n e^{\frac{x^2}{2}} D_x^n e^{-\frac{x^2}{2}} \quad (11.2)$$

$$H_n(x) = (-1)^n e^{x^2} D_x^n e^{-x^2} \quad (11.3)$$

$$H_n(x) = (-1)^n (D_x - 2x)^n 1 \quad (11.4)$$

11.1.3 Explicit Formula for $H_n(x)$

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{n-k}{k} k! (2x)^{n-2k} \quad (11.5)$$

11.2 Derivatives of $H_n(x)$

$$D_x^r H_n(x) = r! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{n-k}{k} k! 2^{n-2k} \binom{n-2k}{r} x^{n-2k-r} \quad (11.6)$$

$$D_x^r H_n(x)|_{x=0} = (-1)^{\frac{n-r}{2}} 2^r \frac{n!}{\left(\frac{n-r}{2}\right)!} \frac{1 + (-1)^{n-r}}{2} \quad (11.7)$$

11.2.1 Taylor's Formula for $H_n(x)$

$$H_n(x) = \sum_{k=0}^n x^k \frac{(-1)^{\frac{n-k}{2}} 2^k n!}{k! \left(\frac{n-k}{2}\right)!} \cdot \frac{1 + (-1)^{n-k}}{2} \quad (11.8)$$

11.3 Operator Analysis with Hermite Polynomials

$$e^{\frac{x^2}{2}} (x - D_x)^n \left(e^{-\frac{x^2}{2}} f(x) \right) = (-1)^n (D_x - 2x)^n f(x) \quad (11.9)$$

$$\begin{aligned} \left(e^{x^2} D_x e^{-x^2} \right)^n f(x) &= e^{x^2} D_x^n \left(e^{-x^2} f(x) \right) = (D_x - 2x)^n f(x) \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} H_k(x) D_x^{n-k} f(x) \end{aligned} \quad (11.10)$$

11.3.1 Recurrence Relation for $H_n(x)$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1 \quad (11.11)$$

11.4 Additional Properties of $H_n(x)$

$$e^{-t^2 D_x^2} x^n = t^n H_n\left(\frac{x}{2t}\right) \quad (11.12)$$

$$(2x)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(n-2k)!} H_{n-2k}(x) \quad (11.13)$$

$$D_x^2 H_n(x) - 2x D_x H_n(x) + 2n H_n(x) = 0 \quad (11.14)$$

$$\frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_k(x) dx = \begin{pmatrix} 0 \\ n-k \end{pmatrix} \quad (11.15)$$

$$H_n(x) = (-1)^n \frac{e^{x^2} n!}{2\pi i} \int_C \frac{e^{-z^2}}{(z-x)^{n+1}} dz, \quad C \text{ is a simple closed contour around } x \quad (11.16)$$

12 Bernstein Polynomials

Remark 12.1 Consider the function f defined on $0 \leq x \leq 1$. The expansion

$$B_n^f(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad n \geq 1 \quad (12.1)$$

is the Bernstein Polynomial of order n for f . A reference for these polynomials is Bernstein's "Démonstration du théorème de Weierstrass, fondée sur le calcul des probabilités", *Commun. Soc. Math Kharkow* (2), Vol. 13, 1912-13, pp. 1-2.

12.1 Connections with Difference Operators

Remark 12.2 For this section, we assume r is a nonnegative integer. We also use the convention that $D_x^r f(x)$ represents the r^{th} derivative of $f(x)$. Finally, recall the definition of Δ_h^r , namely

$$\Delta_h^r f(x) = \sum_{k=0}^r (-1)^{r+k} \binom{r}{k} \frac{f(x+kh)}{h^r}.$$

$$D_x^r B_n^f(x) = r! \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k} x^k (1-x)^{n-r-k} \Delta_1^r f\left(\frac{k}{n}\right) \quad (12.2)$$

$$B_n^f(x) = \sum_{j=0}^n x^j \binom{n}{j} \Delta_1^j f\left(\frac{k}{n}\right) \Big|_{k=0} \quad (12.3)$$

12.1.1 Examples of Equation (12.3)

Let $f(x) = x^r$. Then,

$$B_n^f(x) = \frac{1}{n^r} \sum_{k=0}^n \binom{n}{k} x^k \sum_{j=0}^k (-1)^{j+k} \binom{k}{j} j^r. \quad (12.4)$$

Let $f(x) = e^x$. Then,

$$B_n^f(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} e^{\frac{k}{n}} = \sum_{k=0}^n \binom{n}{k} \left(e^{\frac{1}{n}} - 1\right)^k x^k = \left(1 + \left(e^{\frac{1}{n}} - 1\right)\right)^n. \quad (12.5)$$

13 A Property Catalog for Special Functions

Remark 13.1 In this chapter, except for Section 13.7, we assume n and k are nonnegative integers. We always assume x , t and z are real or complex numbers. We also use the convention that D_x^n denotes the n^{th} derivative with respect to x .

13.1 Laguerre Polynomials $L_n(x)$

Generating Function

$$\frac{e^{\frac{-xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n, \quad |t| < 1 \quad (13.1)$$

Operator Definition

$$L_n(x) = \frac{e^x}{n!} D_x^n (e^{-x} x^n) \quad (13.2)$$

Explicit Formula

$$L_n(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^k}{k!} \quad (13.3)$$

Orthogonality Property

$$\int_0^{\infty} e^{-x} L_n(x) L_k(x) dx = \begin{pmatrix} 0 \\ n-k \end{pmatrix} \quad (13.4)$$

13.2 Generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$

Remark 13.2 Throughout this section, we assume α is a real or complex number independent of n . If $\alpha = 0$, we obtain $L_n(x)$, the Laguerre polynomial of the previous section.

Generating Function

$$\frac{e^{\frac{-xt}{1-t}}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n, \quad |t| < 1 \quad (13.5)$$

Operator Definition

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} D_x^n (e^{-x}x^{\alpha+n}) \quad (13.6)$$

Explicit Formula

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \quad (13.7)$$

Remark 13.3 Recall that, for nonnegative integers p and q , we define

$${}_pF_q[a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k, \dots, (a_p)_k}{k! (b_1)_k (b_2)_k \dots (b_q)_k} z^k, \quad (13.8)$$

where $(a)_k = (a)(a+1)(a+2)\dots(a+k-1)$ whenever k is a positive integer. Otherwise $(a)_0 \equiv 1$. Note that

$$(a)_k = (-1)^k \binom{-a}{k} k!. \quad (13.9)$$

Hypergeometric Series

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1[-n; 1+\alpha; x] \quad (13.10)$$

Recurrence Formula

$$(n+1)L_{n+1}^{(\alpha)}(x) = (-x+2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \quad n \geq 1, \quad (13.11)$$

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1$$

Contour Integral

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{2\pi i} \int_C \frac{z^{n+\alpha} e^{-z}}{(z-x)^{n+1}} dz, \quad x \neq 0, \quad (13.12)$$

C a simple closed curve around $z = x$, not containing $z = 0$

13.3 Tschebyscheff Polynomials

13.3.1 Tschebyscheff Polynomials of the First Kind $T_n(x)$

Explicit Formula

$$T_n(x) = \frac{1}{2^{n-1}} \cos n\theta, \text{ where } x = \cos \theta \quad (13.13)$$

Orthogonality Property

$$\int_{-1}^1 \frac{T_n(x)T_k(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & k \neq n \\ \frac{\pi}{2^{2n-1}}, & k = n \end{cases} \quad (13.14)$$

Operator Definition

$$T_n(x) = \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot \dots (2n-1)} (1-x^2)^{\frac{1}{2}} D_x^n (1-x^2)^{n-\frac{1}{2}} \quad (13.15)$$

13.3.2 Tschebyscheff Polynomials of the Second Kind $U_n(x)$

Explicit Formula

$$U_n(x) = \frac{1}{n+1} D_x T_{n+1}(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \text{ where } x = \cos \theta \quad (13.16)$$

Orthogonality Property

$$\int_{-1}^1 U_n(x)U_k(x)\sqrt{1-x^2} dx = \begin{cases} 0, & k \neq n \\ \frac{\pi}{2}, & k = n \end{cases} \quad (13.17)$$

13.4 Gegenbauer Polynomials $P_n^{(\gamma)}(x)$

Remark 13.4 Throughout this section, we assume γ is a real or complex number independent of n . We also let, whenever x is a real number, $[x]$ denote the floor of x . Finally, we use the Γ function to evaluate noninteger factorials.

Generating Function

$$(1-2xt+t^2)^{-\gamma} = \sum_{n=0}^{\infty} P_n^{(\gamma)}(x)t^n \quad (13.18)$$

Explicit Formulas

$$P_n^{(\gamma)}(x) = \frac{2^{-n}(\gamma - \frac{1}{2})!(n + 2\gamma - 1)!}{(2\gamma - 1)!(n + \gamma - \frac{1}{2})!} \sum_{k=0}^n \binom{n + \gamma - \frac{1}{2}}{n - k} \binom{n + \gamma - \frac{1}{2}}{k} (x - 1)^k (x + 1)^{n-k} \quad (13.19)$$

$$P_n^{(\gamma)}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{\gamma + n - k - 1}{n - k} \binom{n - k}{k} (2x)^{n-2k} \quad (13.20)$$

Hypergeometric Series

$$P_n^{(\gamma)}(x) = \binom{2\gamma + n - 1}{n} {}_2F_1[-n, 2\gamma + n; \gamma + \frac{1}{2}; \frac{1-x}{2}] \quad (13.21)$$

13.4.1 Connections to Other Special Functions

Legendre Polynomials $P_n(x)$

$$P_n(x) = P_n^{(\frac{1}{2})}(x) \quad (13.22)$$

Tschheyscheff Polynomials $U_n(x)$

$$U_n(x) = P_n^{(1)}(x) \quad (13.23)$$

13.5 Jacobi Polynomials $P_n^{(\alpha, \beta)}(x)$

Remark 13.5 Throughout this section, we assume α and β are real or complex numbers independent of n .

Operator Definition

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} D_x^n ((1-x)^{\alpha+n} (1+x)^{\beta+n}) \quad (13.24)$$

Explicit Formulas

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \binom{n + \alpha}{k} \binom{n + \beta}{n - k} \left(\frac{x - 1}{2}\right)^{n-k} \left(\frac{x + 1}{2}\right)^k \quad (13.25)$$

$$(z - 1)^n P_n^{(\alpha, \beta)}\left(\frac{z + 1}{z - 1}\right) = \sum_{k=0}^n \binom{n + \alpha}{k} \binom{n + \beta}{n - k} z^k \quad (13.26)$$

13.5.1 Connections to Other Special Functions

Legendre Polynomials $P_n(x)$

$$P_n(x) = P_n^{(0,0)}(x) \quad (13.27)$$

Ultraspherical Polynomials: The Ultraspherical Polynomials are $P_n^{(\alpha,\alpha)}(x) \equiv P_n^{(\alpha)}$.

Tschebyscheff Polynomials

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(\frac{1}{2}, \frac{1}{2})}(x) \quad (13.28)$$

$$U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) \quad (13.29)$$

13.6 Bessel Polynomials $y_n(x)$

Generating Function

$$e^{\frac{1-(1-2xt)^{\frac{1}{2}}}{x}} = \sum_{n=0}^{\infty} y_{n-1}(x) \frac{t^n}{n!} \quad (13.30)$$

Operator Definition

$$y_n(x) = 2^{-n} e^{\frac{2}{x}} D_x^n \left(x^{2n} e^{-\frac{2}{x}} \right) \quad (13.31)$$

Explicit Formula

$$y_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{(n+k)!}{n!} \left(\frac{x}{2}\right)^k \quad (13.32)$$

Recurrence Relation

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x), \quad n \geq 1 \quad (13.33)$$

Orthogonality Property

$$\frac{1}{2\pi i} \int_{|z|=1} y_n(z)y_k(z)e^{-\frac{2}{z}} dz = \begin{cases} 0, & n \neq k \\ \frac{2(-1)^{n+1}}{2n+1}, & k = n \end{cases} \quad (13.34)$$

Expansion Properties

$$x^n = \sum_{k=0}^n (-1)^k \frac{n!(2n-2k+1)}{k!(2n-k+1)!} 2^n y_{n-k}(x) \quad (13.35)$$

$$e^z = \sum_{n=0}^{\infty} y_{n-1}(x) \frac{(z - \frac{xz^2}{2})^n}{n!} \quad (13.36)$$

13.6.1 Generalized Bessel Polynomials $y_n(x, a, b)$

Explicit Formula

$$y_n(x, a, b) = \sum_{k=0}^n \binom{n}{k} \frac{(n+k+a-2)!}{(n+a-2)!} \left(\frac{x}{b}\right)^k, \quad (13.37)$$

a and b are nonzero real or complex numbers

$$y_n(x) = y_n(x, 2, 2) \quad (13.38)$$

13.7 Ordinary Bessel Functions $J_n(x)$

Remark 13.6 For this section, the subscript of the Bessel Function can be an arbitrary real or complex number. We will specify any restrictions on the subscripts as necessary. We evaluate any noninteger factorials by the Γ function.

13.7.1 Generating Functions

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \quad (13.39)$$

$$J_0(\sqrt{x^2 - 2xt}) = \sum_{n=0}^{\infty} J_n(x) \frac{t^n}{n!} \quad (13.40)$$

13.7.2 Explicit Formulas

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \quad (13.41)$$

$$J_{-n}(x) = (-1)^n J_n(x), \quad n \text{ an integer} \quad (13.42)$$

$$J_n(x) = \frac{1}{2\pi} \int_{\alpha}^{2\pi+\alpha} \cos(nt - x \sin t) dt = \frac{1}{\pi} \int_0^{\pi} \cos(nt - x \sin t) dt, \quad n \text{ an integer} \quad (13.43)$$

$$J_n(z) = \frac{\left(\frac{z}{2}\right)^n}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)} \int_0^{\pi} e^{iz \cos x} \sin^{2n} x dx = \frac{\left(\frac{z}{2}\right)^n}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)} \int_{-1}^1 e^{izt} (1-t^2)^{n-\frac{1}{2}} dt \quad (13.44)$$

13.7.3 Recurrence Formulas

$$2nJ_n(x) = xJ_{n-1}(x) + xJ_{n+1}(x) \quad (13.45)$$

$$xD_x J_n(x) = nJ_n(x) - xJ_{n+1}(x) \quad (13.46)$$

$$xD_x J_n(x) = -nJ_n(x) + xJ_{n-1}(x) \quad (13.47)$$

13.7.4 Operator Properties

$$D_x (x^n J_n(x)) = x^n J_{n-1}(x) \quad (13.48)$$

$$D_x (x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x) \quad (13.49)$$

$$x^{-n-\alpha} J_{n+\alpha}(x) = (-1)^\alpha \left(\frac{1}{x} D_x\right)^\alpha (x^{-n} J_n(x)), \quad \alpha \text{ a nonnegative integer} \quad (13.50)$$

13.7.5 Orthogonality Properties

$$\int_0^{\infty} \frac{J_n(x) J_k(x)}{x} dx = \begin{cases} 0, & k \neq n, \quad k+n > 0 \\ \frac{1}{2n}, & k = n, \quad n > 0 \end{cases} \quad (13.51)$$

Let γ_i be the positive roots of $J_n(x) = 0$. Then,

$$\int_0^1 \sqrt{x} J_n(\gamma_i x) \sqrt{x} J_n(\gamma_j x) dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2} (J_n'(\gamma_i))^2, & i = j. \end{cases} \quad (13.52)$$

13.7.6 Convolution Properties

$$J_n(z)J_m(z) = \sum_{k=0}^{\infty} (-1)^k \binom{n+m+2k}{k} \frac{\left(\frac{z}{2}\right)^{n+m+2k}}{(n+k)!(m+k)!} \quad (13.53)$$

$$J_n(z+x) = \sum_{k=-\infty}^{\infty} J_k(z)J_{n-k}(x) \quad (13.54)$$

Remark 13.7 The following identity is from Harry Bateman. Please see *Proc. of the London Math. Society*, (2), III (1905), p. 120.

$$\frac{J_{m+n}(z)}{m} = \int_0^z \frac{J_m(t)J_n(z-t)}{t} dt, \text{ for } \operatorname{Re}(m) > 0 \text{ and } \operatorname{Re}(n) > -1 \quad (13.55)$$

13.7.7 Connections to Other Special Functions

Legendre Polynomials $P_n(x)$

$$e^{tx} J_0(t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \quad (13.56)$$

Laguerre Polynomials $L_n(x)$

$$e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!} \quad (13.57)$$

Combinatorial Identities: Table III: Binomial Identities Derived from Trigonometric and Exponential Series

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 Basic Trigonometric Series

Remark 1.1 *Throughout this chapter, we assume n and a are nonnegative integers. We assume x and y are real or complex numbers.*

1.1 Telescoping Trigonometric Series

$$\sum_{k=1}^n \sin \frac{2k+1}{2}x = \frac{\cos(n+1)x - \cos x}{-2 \sin \frac{x}{2}}, \quad n \geq 1 \quad (1.1)$$

$$\sum_{k=1}^n \sin \frac{2k+1}{2}x = \frac{\sin \frac{(n+2)x}{2} \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}, \quad n \geq 1 \quad (1.2)$$

$$\sum_{k=1}^n \cos \frac{2k+1}{2}x = \frac{\sin(n+1)x - \sin x}{2 \sin \frac{x}{2}}, \quad n \geq 1 \quad (1.3)$$

$$\sum_{k=1}^n \cos \frac{2k+1}{2}x = \frac{\cos \frac{(n+2)x}{2} \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}, \quad n \geq 1 \quad (1.4)$$

$$\sum_{k=1}^n \frac{\sin \frac{k}{(k+1)!}x}{\cos \frac{x}{k!} \cdot \cos \frac{x}{(k+1)!}} = \tan x - \tan \frac{x}{(n+1)!}, \quad n \geq 1 \quad (1.5)$$

$$\sum_{k=1}^{\infty} \frac{\sin \frac{k}{(k+1)!}x}{\cos \frac{x}{k!} \cdot \cos \frac{x}{(k+1)!}} = \tan x \quad (1.6)$$

$$\sum_{k=1}^n \frac{\tan 2^k x}{\cos 2^{k+1} x} = \tan 2^{n+1} x - \tan 2x, \quad n \geq 1 \quad (1.7)$$

$$\sum_{k=1}^n \sec(k+1)x \cdot \sec kx = \frac{\tan(n+1)x - \tan x}{\sin x}, \quad n \geq 1 \quad (1.8)$$

$$\sum_{k=1}^n \sin(x + (k-1)y) = \frac{\sin(x + \frac{n-1}{2}y) \cdot \sin \frac{ny}{2}}{\sin \frac{y}{2}}, \quad n \geq 1 \quad (1.9)$$

$$\sum_{k=1}^n \sin kx = \frac{\sin \frac{(n+1)x}{2} \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}, \quad n \geq 1 \quad (1.10)$$

$$\sum_{k=1}^n \sin(2k-1)x = \frac{\sin^2 nx}{\sin x}, \quad n \geq 1 \quad (1.11)$$

$$\sum_{k=1}^n k \cos \frac{(2k+1)x}{2} = \frac{(n+1) \sin(n+1)x \cdot \sin \frac{x}{2} - \sin \frac{(n+2)x}{2} \cdot \sin \frac{(n+1)x}{2}}{2 \sin^2 \frac{x}{2}} \quad (1.12)$$

$$\sum_{k=1}^n \cos(x + (k-1)y) = \frac{\cos(x + \frac{n-1}{2}y) \cdot \sin \frac{ny}{2}}{\sin \frac{y}{2}}, \quad n \geq 1 \quad (1.13)$$

$$\sum_{k=1}^n \cos kx = \frac{\cos \frac{(n+1)x}{2} \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}, \quad n \geq 1 \quad (1.14)$$

$$\sum_{k=1}^n \cos(2k-1)x = \frac{\sin 2nx}{2 \sin x}, \quad n \geq 1 \quad (1.15)$$

$$\sum_{k=0}^n \cos^3(x + ky) = \frac{\cos(3x + \frac{3ny}{2}) \sin(\frac{3y(n+1)}{2})}{4 \sin \frac{3y}{2}} + \frac{3 \cos(x + \frac{ny}{2}) \sin \frac{(n+1)y}{2}}{4 \sin \frac{y}{2}} \quad (1.16)$$

1.2 Sums and Products Based on Double Angle Formulas

$$\prod_{k=0}^n \cos 2^k x = \frac{\sin 2^{n+1} x}{2^{n+1} \sin x} \quad (1.17)$$

$$\prod_{k=1}^n \cos \frac{x}{2^k} = \frac{\sin x}{2^n \sin \frac{x}{2^n}}, \quad n \geq 1 \quad (1.18)$$

$$\prod_{k=1}^{\infty} \cos \frac{x}{2^k} = \frac{\sin x}{x} \quad (1.19)$$

$$\sum_{k=1}^n \sin^2 kx = \frac{n}{2} - \frac{\cos(n+1)x \cdot \sin nx}{2 \sin x}, \quad n \geq 1 \quad (1.20)$$

$$\sum_{k=1}^n \cos^2 kx = \frac{n}{2} + \frac{\cos(n+1)x \cdot \sin nx}{2 \sin x}, \quad n \geq 1 \quad (1.21)$$

1.3 Sums Based on Half Angle Formulas

$$\sum_{k=1}^n \csc 2^{k-1} x = \cot \frac{x}{2} - \cot 2^{n-1} x, \quad n \geq 1 \quad (1.22)$$

$$\sum_{k=0}^n \csc \frac{x}{2^k} = \cot \frac{x}{2^{n+1}} - \cot x \quad (1.23)$$

$$\sum_{k=1}^n \frac{1}{2^{k-1}} \tan \frac{x}{2^{k-1}} = \frac{1}{2^{n-1}} \cot \frac{x}{2^{n-1}} - 2 \cot 2x, \quad n \geq 1 \quad (1.24)$$

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \tan \frac{x}{2^{k-1}} = \frac{1}{x} - 2 \cot 2x \quad (1.25)$$

1.4 Miscellaneous Trigonometric Series

$$\sum_{k=a}^{n-1} 3^k \sin^3 \frac{x}{3^{k+1}} = \frac{1}{4} \left(3^n \sin \frac{x}{3^n} - 3^a \sin \frac{x}{3^a} \right), \quad n \geq 1 \quad (1.26)$$

$$\sum_{k=a}^{n-1} (-3)^k \cos^3 \frac{x}{3^{k+1}} = \frac{1}{4} \left((-3)^a \cos \frac{x}{3^a} - (-3)^n \cos \frac{x}{3^n} \right), \quad n \geq 1 \quad (1.27)$$

2 The Exponential Function with Trigonometric Series

Remark 2.1 Throughout this chapter, we assume n and r are nonnegative integers. We let x , y , and z denote real or complex numbers. Furthermore, we reserve $i \equiv \sqrt{-1}$. Finally if x is a real number, we let $[x]$ denote the floor of x .

2.1 Limit Definition for e

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = \lim_{n \rightarrow \infty} \sum_{r=0}^n \binom{n}{r} \left(\frac{1}{n} \right)^r \quad (2.1)$$

2.2 The Exponential Series and Various Applications

2.2.1 The Exponential Series

$$e^z = \sum_{r=0}^{\infty} \frac{z^r}{r!} \quad (2.2)$$

2.2.2 Series from $e^{ix} = \cos x + i \sin x$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x \quad (2.3)$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin x \quad (2.4)$$

$$\sum_{n=0}^{\infty} \frac{x^n \cos ny}{n!} = e^{x \cos y} \cos(x \sin y) \quad (2.5)$$

$$\sum_{n=0}^{\infty} \frac{x^n \sin ny}{n!} = e^{x \cos y} \sin(x \sin y) \quad (2.6)$$

Remark 2.2 *The following identity is from Problem 415 of The Mathematics Magazine, May 1960. Solutions to this problem are found in The Mathematics Magazine, Vol. 34, No. 3, 1961, P. 178.*

$$\sum_{k=0}^n \binom{n}{k} \cos kx \cdot \sin(n-k)x = 2^{n-1} \sin nx \quad (2.7)$$

$$\cos^n x \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \tan^{2k} x = \cos nx \quad (2.8)$$

$$\cos^n x \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \tan^{2k+1} x = \sin nx, \quad n \geq 1 \quad (2.9)$$

2.3 Expansions of $(e^{ix} \pm 1)^n$

2.3.1 Expansions of $(e^{ix} + 1)^n$

$$\sum_{k=0}^n \binom{n}{k} \cos kx = 2^n \cos \frac{nx}{2} \left(\cos \frac{x}{2} \right)^n \quad (2.10)$$

$$\sum_{k=0}^n \binom{n}{k} \sin kx = 2^n \sin \frac{nx}{2} \left(\cos \frac{x}{2} \right)^n \quad (2.11)$$

2.3.2 Expansions of $(e^{ix} - 1)^n$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \cos kx = (-2)^n \cos \frac{n(x+\pi)}{2} \left(\sin \frac{x}{2} \right)^n \quad (2.12)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \sin kx = (-2)^n \sin \frac{n(x+\pi)}{2} \left(\sin \frac{x}{2}\right)^n \quad (2.13)$$

Inversion of Identity (2.12)

$$\sum_{k=1}^n (-1)^{k-1} 2^k \left(\sin \frac{x}{2}\right)^k \cos \frac{k(x+\pi)}{2} + n = \sum_{k=1}^n (-1)^{k-1} \binom{n+1}{k+1} \cos kx \quad (2.14)$$

2.3.3 Applications of Equations (2.10), (2.11), (2.12), and (2.13)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \cos kx = 2^{n-1} \left(\cos^n \left(\frac{x}{4}\right) \cos \left(\frac{nx}{4}\right) + (-1)^n \sin^n \left(\frac{x}{4}\right) \cos \left(\frac{n\pi}{2} + \frac{nx}{4}\right) \right) \quad (2.15)$$

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \cos(2k+1)x &= 2^{n-1} \cos^n \left(\frac{x}{2}\right) \cos \left(\frac{nx}{2}\right) \\ &\quad - 2^{n-1} (-1)^n \sin^n \left(\frac{x}{2}\right) \cos \left(\frac{n(\pi+x)}{2}\right), \quad n \geq 1 \end{aligned} \quad (2.16)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \sin kx = 2^{n-1} \left(\cos^n \left(\frac{x}{4}\right) \sin \left(\frac{nx}{4}\right) + (-1)^n \sin^n \left(\frac{x}{4}\right) \sin \left(\frac{n\pi}{2} + \frac{nx}{4}\right) \right) \quad (2.17)$$

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \sin(2k+1)x &= 2^{n-1} \cos^n \left(\frac{x}{2}\right) \sin \left(\frac{nx}{2}\right) \\ &\quad - 2^{n-1} (-1)^n \sin^n \left(\frac{x}{2}\right) \sin \left(\frac{n(\pi+x)}{2}\right), \quad n \geq 1 \end{aligned} \quad (2.18)$$

2.4 The Geometric Series $\sum_{k=1}^n (ye^{ix})^k$

$$\sum_{k=1}^n y^k \cos kx = \frac{y^{n+2} \cos nx - y^{n+1} \cos(n+1)x + y \cos x - y^2}{y^2 - 2y \cos x + 1}, \quad n \geq 1 \quad (2.19)$$

$$\sum_{k=1}^n y^k \sin kx = \frac{y^{n+2} \sin nx - y^{n+1} \sin(n+1)x + y \sin x}{y^2 - 2y \cos x + 1}, \quad n \geq 1 \quad (2.20)$$

$$\sum_{k=1}^{\infty} y^k \cos kx = \frac{y \cos x - y^2}{y^2 - 2y \cos x + 1}, \quad |y| < 1 \quad (2.21)$$

$$\sum_{k=1}^{\infty} y^k \sin kx = \frac{y \sin x}{y^2 - 2y \cos x + 1}, \quad |y| < 1 \quad (2.22)$$

$$\sum_{k=1}^{\infty} \frac{y^k \cos kx}{k} = \frac{1}{2} \ln \frac{1}{1 - 2y \cos x + y^2}, \quad |y| < 1, \quad (2.23)$$

if y is a complex number, use the principle value of $\ln y$

$$\sum_{k=1}^{\infty} \frac{y^k \sin kx}{k} = \arctan \frac{y \sin x}{1 - y \cos x}, \quad |y| < 1 \quad (2.24)$$

3 Advanced Trigonometric Series Expansions

Remark 3.1 Throughout this chapter, we assume n and j are nonnegative integers, while x and y are real or complex numbers. We also let $[x]$ denote the floor of x (for real x).

3.1 Two Identities Associated with Coefficients in Trigonometric Expansions

3.1.1 First Identity

$$\sum_{k=j}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} \binom{k}{j} = 2^{n-2j} \binom{n-j}{j}, \quad j \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (3.1)$$

Applications of Equation (3.1)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} = 2^n \quad (3.2)$$

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} k = (n-1)2^{n-2}, \quad n \geq 1 \quad (3.3)$$

$$\sum_{k=0}^n \binom{4n+1}{2n-2k} \binom{k+n}{n} = 2^{2n} \binom{3n}{n} \quad (3.4)$$

3.1.2 Second Identity

$$\sum_{k=j}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \binom{k}{j} = 2^{n-2j} \binom{n-j}{j} - 2^{n-1-2j} \binom{n-1-j}{j}, \quad j \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (3.5)$$

Restatement of Equation (3.5)

$$\sum_{k=j}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \binom{k}{j} = \frac{n2^{n-2j-1}}{n-j} \binom{n-j}{j}, \quad j \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (3.6)$$

Applications of Equation (3.5)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1} \quad (3.7)$$

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} k = n2^{n-2}, \quad n \geq 2 \quad (3.8)$$

Applications of Equation (3.6)

$$\sum_{k=0}^n \binom{4n}{2n-2k} \binom{k+n}{n} = \frac{2^{2n+1}}{3} \binom{3n}{n} \quad (3.9)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{x^k}{n-k} = \frac{(1 + \sqrt{4x+1})^n + (1 - \sqrt{4x+1})^n}{n2^n}, \quad n \geq 1 \quad (3.10)$$

Applications of Equation (3.10)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{1}{n-k} = \frac{(1 + \sqrt{5})^n + (1 - \sqrt{5})^n}{n2^n}, \quad n \geq 1 \quad (3.11)$$

Remark 3.2 The following identity is equivalent of Example 44, p. 445 of Hardy's Pure Mathematics.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{1}{n-k} = \begin{cases} (-1)^{n-1} \frac{1}{n}, & \text{if } n \text{ is not a multiple of } 3 \\ (-1)^{\frac{n}{2}}, & \text{if } n \text{ is a multiple of } 3 \end{cases} \quad (3.12)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{6^k}{n-k} = \frac{3^n + (-1)^n 2^n}{n}, \quad n \geq 1 \quad (3.13)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{4^{n-k}}{n-k} = \frac{2^{n+1}}{n}, \quad n \geq 1 \quad (3.14)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{4^{n-k}}{k+1} = \frac{4^{n+1} - 2^{n+1}}{n+2} \quad (3.15)$$

3.2 Expansion of $\frac{\sin(n+1)x}{\sin x}$

$$\frac{\sin(n+1)x}{\sin x} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2 \cos x)^{n-2k} \quad (3.16)$$

3.3 Expansion of $\cos nx$

$$\cos nx = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \cos^{n-2k} x \left(2^{n-2k} \binom{n-k}{k} - 2^{n-1-2k} \binom{n-k-1}{k} \right) \quad (3.17)$$

$$\cos nx = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{(2 \cos x)^{n-2k}}{n-k}, \quad n \geq 1 \quad (3.18)$$

3.4 Expansions of $\cos 2nx$

3.4.1 Using $\sin^{2k} x$

$$\cos 2nx = \sum_{k=0}^n (-1)^k \sin^{2k} x \sum_{j=0}^k \binom{2n}{2j} \binom{n-k}{k-j} \quad (3.19)$$

$$\sum_{j=0}^k \binom{2n}{2j} \binom{n-j}{k-j} = \frac{2^{2k}}{(2k)!} \prod_{j=0}^{k-1} (n^2 - j^2) = \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} \quad (3.20)$$

Restatement of Equation (3.19)

$$\cos 2nx = \sum_{k=0}^n (-1)^k \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} \sin^{2k} x, \quad n \geq 1 \quad (3.21)$$

3.4.2 Using $\cos^{2k} x$

$$\cos 2nx = \sum_{k=0}^n (-1)^{n-k} \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} \cos^{2k} x, \quad n \geq 1 \quad (3.22)$$

3.4.3 Binomial Identities Resulting From the Coefficient of $\cos^{2k} x$ in Equation (3.22)

$$\sum_{k=j}^n (-1)^k \binom{k}{j} \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} = (-1)^n \frac{n}{n+j} \binom{n+j}{2j} 2^{2j}, \quad n \geq 1 \quad (3.23)$$

$$\sum_{k=0}^n (-1)^k \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} = (-1)^n, \quad n \geq 1 \quad (3.24)$$

$$\sum_{k=1}^n (-1)^k \frac{2^{2k}}{(2k)!} \prod_{j=0}^{k-1} (n^2 - j^2) = (-1)^n - 1, \quad n \geq 1 \quad (3.25)$$

Generalization of Equation (3.20)

$$\sum_{k=0}^n \binom{2x}{2k} \binom{x-k}{n-k} = \frac{2^{2n}}{(2n)!} \prod_{k=0}^{n-1} (n^2 - k^2) = \frac{x}{x+n} \binom{x+n}{2n} 2^{2n} \quad (3.26)$$

3.4.4 Applications of Equation (3.26)

$$\sum_{k=0}^n (-1)^k \binom{2n}{n-k} \binom{2n+2k+1}{2k} = (-1)^n (n+1) 2^{2n} \quad (3.27)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{1-2n} \quad (3.28)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+k}{k} \frac{2^{2k}}{\binom{2k}{k} (x+k)} = (-1)^n \frac{\binom{2x}{2n}}{x \binom{x}{n}}, \quad x \neq 0 \quad (3.29)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{2^{2k}}{\binom{2k}{k} (n+k)} = \frac{(-1)^n}{n}, \quad n \geq 1 \quad (3.30)$$

n^{th} Difference of the Harmonic Series

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{j=1}^{2k} \frac{1}{j} = \frac{1}{2n} + \frac{2^{2n-1}}{n \binom{2n}{n}}, \quad n \geq 1 \quad (3.31)$$

Inversion of Equation (3.31)

$$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{2^{2k}}{k \binom{2k}{k}} = 2 \sum_{j=1}^{2n} \frac{1}{j} - \sum_{j=1}^n \frac{1}{j}, \quad n \geq 1 \quad (3.32)$$

3.5 Expansions of $\frac{\sin(2n+1)x}{\sin x}$

3.5.1 Using $\sin^{2k} x$

$$\sin(2n+1)x = \sum_{k=0}^n (-1)^k \sin^{2k+1} x \sum_{j=0}^k \binom{2n+1}{2j+1} \binom{n-j}{k-j} \quad (3.33)$$

$$\begin{aligned} \sum_{j=0}^k \binom{2n+1}{2j+1} \binom{n-j}{k-j} &= \frac{2n+1}{(2k+1)!} \prod_{j=0}^{k-1} ((2n+1)^2 - (2j+1)^2) \\ &= 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} \end{aligned} \quad (3.34)$$

$$\begin{aligned} \sum_{j=0}^k \binom{2n+1}{2k-2j+1} \binom{n-k+j}{j} &= \frac{2n+1}{(2k+1)!} \prod_{j=0}^{k-1} ((2n+1)^2 - (2j+1)^2) \\ &= 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} \end{aligned} \quad (3.35)$$

Restatement of Equation (3.33)

$$\sin(2n+1)x = \sum_{k=0}^n (-1)^k 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} \sin^{2k+1} x \quad (3.36)$$

3.5.2 Using $\cos^{2k} x$

$$\frac{\sin(2n+1)x}{\sin x} = \sum_{k=0}^n (-1)^{n-k} \cos^{2k} x \sum_{j=0}^k \binom{2n+1}{2k-2j} \binom{n-k+j}{j} \quad (3.37)$$

$$\frac{\sin(2n+1)x}{\sin x} = \sum_{k=0}^n (-1)^{n-k} \cos^{2k} x \sum_{j=0}^k \binom{2n+1}{2j} \binom{n-j}{k-j} \quad (3.38)$$

3.5.3 Binomial Identities Resulting From Coefficient of $\sin^{2k} x$ in Equation (3.36)

$$\sum_{k=j}^n (-1)^k 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} \binom{n}{j} = (-1)^n \sum_{r=0}^j \binom{2n+1}{2r} \binom{n-r}{j-r} \quad (3.39)$$

$$\sum_{k=0}^n (-1)^k 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} = (-1)^n \quad (3.40)$$

Generalization of Equation (3.34)

$$\begin{aligned} \sum_{k=0}^n \binom{2x+1}{2k+1} \binom{x-k}{n-k} &= \frac{2x+1}{(2n+1)!} \prod_{k=0}^{n-1} ((2x+1)^2 - (2k+1)^2) \\ &= 2^{2n} \frac{2x+1}{2n+1} \binom{x+n}{2n} \end{aligned} \quad (3.41)$$

3.5.4 Product Expansion for $\frac{\sin(2n+1)x}{\sin x}$

$$\frac{\sin(2n+1)x}{\sin x} = (2n+1) \prod_{k=1}^n \left(1 - \frac{\sin^2 x}{\sin^2 \frac{\pi k}{2n+1}} \right), \quad n \geq 1 \quad (3.42)$$

3.6 Series for $\cos^n x$

$$2^{n-1} \cos^n x = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \cos(n-2k)x - \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^n + 1}{2} \cos x \quad (3.43)$$

3.6.1 Applications of Equation (3.43)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = 2^{n-1} + \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^n + 1}{2} \quad (3.44)$$

$$\cos^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \cos(2n+1-2k)x \quad (3.45)$$

$$\int \cos^{2n+1} x dx = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin(2n+1-2k)x}{2n+1-2k} + C \quad (3.46)$$

$$\cos^{2n} x = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos(2n-2k)x + \frac{1}{2^{2n}} \binom{2n}{n}, \quad n \geq 1 \quad (3.47)$$

$$\int \cos^{2n} x dx = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k} + \frac{1}{2^{2n}} \binom{2n}{n} x + C, \quad n \geq 1 \quad (3.48)$$

$$\frac{\sin(2n+1)x}{\sin x} = 2 \sum_{k=0}^n \cos 2kx - 1 \quad (3.49)$$

3.7 Series for $\sin^n x$

$$2^{n-1} \sin^n x = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \cos \left((n-2k)x - \frac{n-2k}{2} \pi \right) - \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^n + 1}{2} \quad (3.50)$$

3.7.1 Applications of Equation (3.50)

$$\sin^{2n} x = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cos(2n-2k)x + \frac{1}{2^{2n}} \binom{2n}{n}, \quad n \geq 1 \quad (3.51)$$

$$\int \sin^{2n} x dx = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k} + \frac{1}{2^{2n}} \binom{2n}{n} x + C, \quad n \geq 1 \quad (3.52)$$

$$\sin^{2n+1} x = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \sin(2n+1-2k)x \quad (3.53)$$

$$\int \sin^{2n+1} x \, dx = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^n (-1)^{k+1} \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k} + C \quad (3.54)$$

4 Advanced Trigonometric Product Expansions

Remark 4.1 For this chapter, we assume n is a nonnegative integer, while x , y , and z are real or complex numbers. We also assume, that whenever x is a real number, $[x]$ denotes the floor of x .

4.1 Product Expansion of $\cos nx - \cos ny$

$$\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left(\cos x - \cos \left(y + \frac{2k\pi}{n} \right) \right), \quad n \geq 1 \quad (4.1)$$

4.1.1 Applications of Equation (4.1)

$$\cos nx + 1 = 2^{n-1} \prod_{k=0}^{n-1} \left(\cos x - \cos \frac{2k+1}{n}\pi \right), \quad n \geq 1 \quad (4.2)$$

$$\prod_{k=0}^{n-1} \cos \left(y + \frac{2k\pi}{n} \right) = \frac{1}{2^{n-1}} \left((-1)^{[\frac{n}{2}]} \frac{1 + (-1)^n}{2} - (-1)^n \cos ny \right), \quad n \geq 1 \quad (4.3)$$

$$\prod_{k=0}^{2n-1} \cos \left(y + \frac{k\pi}{n} \right) = \frac{(-1)^n - \cos 2ny}{2^{2n-1}}, \quad n \geq 1 \quad (4.4)$$

$$\prod_{k=0}^{2n} \cos \left(y + \frac{2k\pi}{2n+1} \right) = \frac{\cos(2n+1)y}{2^{2n}} \quad (4.5)$$

$$\prod_{k=0}^{2n} \cos \frac{2k\pi}{2n+1} = \frac{1}{2^{2n}} \quad (4.6)$$

$$\prod_{k=0}^{2n} \cos \frac{2k+1}{2n+1} \pi = -\frac{1}{2^{2n}} \quad (4.7)$$

$$\prod_{k=0}^{4n+1} \cos \frac{k\pi}{2n+1} = -\frac{1}{2^{4n}} \quad (4.8)$$

4.1.2 Product Expansion of $\sin nx$

$$\sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left(x + \frac{k\pi}{n} \right), \quad n \geq 1 \quad (4.9)$$

Applications of Equation (4.9)

$$\sin^2 \frac{ny}{2} = 2^{2n-2} \prod_{k=0}^{n-1} \sin^2 \left(\frac{y}{2} + \frac{k\pi}{n} \right), \quad n \geq 1 \quad (4.10)$$

$$\prod_{k=0}^{n-1} \sin \left(\frac{k\pi + x}{n} \right) = \frac{\sin x}{2^{n-1}}, \quad n \geq 1 \quad (4.11)$$

$$\prod_{k=1}^{n-1} \cos \frac{k\pi}{n} = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{2^{n-1}} \left(\frac{1 - (-1)^n}{2} \right), \quad n \geq 2 \quad (4.12)$$

$$\prod_{k=1}^{2n} \cos \frac{k\pi}{2n+1} = \frac{(-1)^n}{2^{2n}}, \quad n \geq 1 \quad (4.13)$$

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}, \quad n \geq 2 \quad (4.14)$$

$$\prod_{k=1}^{n-1} \Gamma \left(\frac{k}{n} \right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}, \quad n \geq 1 \quad (4.15)$$

$$\prod_{k=1}^{n-1} \cot \frac{k\pi}{n} = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{n} \left(\frac{1 - (-1)^n}{2} \right), \quad n \geq 2 \quad (4.16)$$

$$n \cot nx = \sum_{k=0}^{n-1} \cot \left(x + \frac{k\pi}{n} \right), \quad n \geq 1 \quad (4.17)$$

$$n^2 \csc^2 nx = \sum_{k=0}^{n-1} \csc^2 \left(x + \frac{k\pi}{n} \right), \quad n \geq 1 \quad (4.18)$$

4.2 Various Product Expansions Involving Equations (4.11) and (4.14)

4.2.1 Expansions Involving Equation (4.14)

$$\prod_{k=1}^{n-1} \left(\sin \frac{k\pi}{n} \right)^k = \frac{\sqrt{n^n}}{2^{\frac{n(n-1)}{2}}}, \quad n \geq 2 \quad (4.19)$$

Remark 4.2 The following identity, proposed by J. E. Wilkins, Jr., is found in Problem E1044 of *The American Math. Monthly*, Vol. 59, No. 10, December 1952.

$$\prod_{k=1}^{n-1} \left(2 \sin \frac{k\pi}{n} \right)^k = \sqrt{n^n}, \quad n \geq 2 \quad (4.20)$$

4.2.2 Expansion Involving Equation (4.11)

$$\prod_{k=1}^{n-1} \left(\sin \frac{k\pi - x}{n} \sin \frac{k\pi + x}{n} \right)^k = \frac{1}{2^{n(n-1)}} \left(\frac{\sin x}{\sin \frac{x}{n}} \right)^n, \quad n \geq 2 \quad (4.21)$$

Applications of Equation (4.21)

$$\prod_{k=1}^{n-1} \left(\cos \frac{k\pi}{n} \right)^{2k} = \frac{(-1)^{\frac{n(n-1)}{2}}}{2^{n(n-1)}} (-1)^{n[\frac{n}{2}]} \left(\frac{1 - (-1)^n}{2} \right)^n, \quad n \geq 2 \quad (4.22)$$

$$n^2 \cot x - n \cot \frac{x}{n} = \sum_{k=1}^{n-1} k \left(\cot \frac{k\pi + x}{n} - \cot \frac{k\pi - x}{n} \right), \quad n \geq 2 \quad (4.23)$$

$$\sum_{k=1}^{n-1} k \csc \frac{k\pi + x}{n} \cdot \csc \frac{k\pi - x}{n} = \frac{n \cot \frac{x}{n} - n^2 \cot x}{\sin \frac{2x}{n}}, \quad n \geq 2 \quad (4.24)$$

4.2.3 Product Expansion for $\tan x$

$$\tan x = \prod_{k=0}^{2n-1} \left(\sin \frac{k\pi + 2x}{2n} \right)^{(-1)^k}, \quad n \geq 1 \quad (4.25)$$

Applications of Equation (4.25)

$$\sum_{k=0}^{2n-1} (-1)^k \cot \frac{k\pi + 2x}{2n} = \frac{2n}{\sin 2x}, \quad n \geq 1 \quad (4.26)$$

$$\sum_{k=0}^{2n-1} (-1)^k \cot \frac{2k+1}{4n} \pi = 2n, \quad n \geq 1 \quad (4.27)$$

$$\sum_{k=0}^{2n-1} (-1)^k \cot \frac{3k+1}{6n} \pi = \frac{4n\sqrt{3}}{3}, \quad n \geq 1 \quad (4.28)$$

$$\sum_{k=0}^{2n-1} (-1)^k \csc^2 \frac{k\pi + 2x}{2n} = 4n^2 \csc 2x \cdot \cot 2x, \quad n \geq 1 \quad (4.29)$$

$$\sum_{k=0}^{2n-1} \cot \frac{4k+1}{4n} \pi = 2n, \quad n \geq 1 \quad (4.30)$$

$$\sum_{k=0}^{n-1} \cot \frac{4k+1}{4n} \pi = n, \quad n \geq 1 \quad (4.31)$$

Remark 4.3 The following identity is Problem 4220 of *The American Math Monthly*, Vol. 58, No.1, May 1952.

$$\sum_{k=0}^{n-1} (-1)^k \tan \frac{2k+1}{4n} \pi = (-1)^{n+1} n, \quad n \geq 1 \quad (4.32)$$

$$\sum_{k=0}^{n-1} \tan \frac{4k+1}{4n} \pi = (-1)^{n+1} n, \quad n \geq 1 \quad (4.33)$$

4.3 Expansions of $\cot z$

$$\cot z = \frac{1}{2n+1} \cot \frac{z}{2n+1} + \sum_{k=1}^n \left(\frac{1}{2n+1} \cot \frac{z+k\pi}{2n+1} + \frac{1}{2n+1} \cot \frac{z-k\pi}{2n+1} \right) \quad (4.34)$$

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z+k\pi} + \frac{1}{z-k\pi} \right), \quad z \text{ not a multiple of } \pi \quad (4.35)$$

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z+k} + \frac{1}{z-k} \right), \quad z \text{ not integral} \quad (4.36)$$

4.3.1 Applications of Equation (4.36)

$$\pi \csc \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2z}{z^2 - k^2}, \quad z \text{ not integral} \quad (4.37)$$

$$\pi^2 \csc^2 \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}, \quad z \text{ not integral} \quad (4.38)$$

$$\pi^3 \cot \pi z \csc^2 \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^3}, \quad z \text{ not integral} \quad (4.39)$$

$$\pi^4 \left(\csc^4 \pi z - \frac{2}{3} \csc^2 \pi z \right) = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^4}, \quad z \text{ not integral} \quad (4.40)$$

$$\pi \tan \frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{4z}{(2k+1)^2 - z^2} \quad (4.41)$$

$$\pi \sec \pi z = \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{\left(\frac{2k+1}{2}\right)^2 - z^2} \quad (4.42)$$

4.4 Expansions of $z \cot z$ via the Bernoulli Numbers

Remark 4.4 In this section, we let \mathcal{B}_n denote the n^{th} Bernoulli number.

$$z \cot z = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} \mathcal{B}_{2k}}{(2k)!} z^{2k}, \quad |z| < \pi \quad (4.43)$$

$$z \cot z = 1 - 2 \sum_{j=1}^{\infty} \frac{z^{2j}}{\pi^{2j}} \sum_{k=1}^{\infty} \frac{1}{k^{2j}}, \quad |z| < \pi \quad (4.44)$$

4.4.1 Applications of Equations (4.43) and (4.44)

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n)!} \mathcal{B}_{2n}, \quad n \geq 1 \quad (4.45)$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = (-1)^{n-1} \frac{(2^{2n}-1) \pi^{2n}}{2(2n)!} \mathcal{B}_{2n}, \quad n \geq 1 \quad (4.46)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} = (-1)^{n-1} \frac{(2^{2n}-1) \pi^{2n}}{(2n)!} \mathcal{B}_{2n}, \quad n \geq 1 \quad (4.47)$$

$$\tan z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k}-1) \mathcal{B}_{2k}}{(2k)!} z^{2k-1}, \quad |z| < \frac{\pi}{2} \quad (4.48)$$

$$\frac{z}{\sin z} = \sum_{k=0}^{\infty} (-1)^k \frac{2(1-2^{2k-1}) \mathcal{B}_{2k}}{(2k)!} z^{2k}, \quad |z| < \pi, \quad z \neq 0 \quad (4.49)$$

5 Series Associated with the Beta Function

Remark 5.1 We assume m, n, k , and r are nonnegative integers, while x, y , and t are real or complex numbers. If necessary, we use the Gamma function to evaluate $x!$ as $\Gamma(x) = (x-1)!$. Finally, recall that $[x]$ denotes the floor of x (for real x).

5.1 Formulas from $\int_0^{\frac{\pi}{2}} \sin^x t \cos^y t dt$

$$\int_0^{\frac{\pi}{2}} \sin^x t \cos^y t dt = \frac{\pi}{2^{x+y+1}} \frac{x!y!}{\left(\frac{x}{2}\right)! \left(\frac{y}{2}\right)! \left(\frac{x+y}{2}\right)!} \quad (5.1)$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x \cos^{2n} x dx = \frac{\pi \binom{2k}{k} \binom{2n}{n}}{2^{2n+2k+1} \binom{n+k}{k}} \quad (5.2)$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{\pi}{2^{n+1}} \binom{n}{\frac{n}{2}} \quad (5.3)$$

5.2 Applications of Equation (5.2)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n+k}{k}} = 1 \quad (5.4)$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \binom{-\frac{1}{2}}{k} \frac{2^{2k}}{\binom{n+k}{k}} = 1 \quad (5.5)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{n+k} \binom{2k}{k} = \binom{3n}{n} \quad (5.6)$$

$$\sum_{k=0}^{2n} \binom{3n}{2n-k} \binom{-\frac{1}{2}}{k} 2^{2k} = \binom{3n}{n} \quad (5.7)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2k}{n+k} 3^{2n-k} = \binom{2n}{n} \quad (5.8)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{z^{2k}}{\binom{m+k}{k}} = \frac{2^{2m+1}}{\pi \binom{2m}{m}} \int_0^{\frac{\pi}{2}} (1 - 4z^2 \sin^2 x)^n \cos^{2m} x \, dx \quad (5.9)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{z^{2k}}{\binom{m+k}{k}} = \frac{2^{2m+1}}{\pi \binom{2m}{m}} \int_0^{\frac{\pi}{2}} (1 - 4z^2 \cos^2 x)^n \sin^{2m} x \, dx \quad (5.10)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n+k}{k} 2^{2k}} = \frac{\binom{6n}{3n}}{2^{4n} \binom{2n}{n}} \quad (5.11)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n+r+k}{k}} = \frac{2^{2n+2r+1}}{\pi \binom{2n+2r}{n+r}} \int_0^{\frac{\pi}{2}} \cos^{2r} x (\cos 3x)^{2n} \, dx \quad (5.12)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n+r+k}{k}} = \frac{2^{2n+2r+1}}{\pi \binom{2n+2r}{n+r}} \int_0^{\frac{\pi}{2}} \sin^{2r} x (\sin 3x)^{2n} \, dx \quad (5.13)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{2^{2k}} = \frac{1}{2^{2n}} \binom{2n}{n} \quad (5.14)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 3^{n-k} \quad (5.15)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{2^{2k} \binom{m+k}{k}} = \frac{\binom{2m+2n}{m+n}}{2^{2n} \binom{2m}{m}} \quad (5.16)$$

5.3 Generalization of Equation (5.12)

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n+r+k}{k}} = \frac{2^{2n+2r+1}}{\pi \binom{2n+2r}{n+r}} \int_0^{\frac{\pi}{2}} \cos^{2r-1} x (\cos 3x)^{2n+1} \, dx \quad (5.17)$$

5.3.1 Various Applications of 5.17 and 5.12

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n+1+k}{k}} = 0 \quad (5.18)$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n+2+k}{k}} = \frac{n+2}{2(2n+3)} \quad (5.19)$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n+k}{k}} = -1 \quad (5.20)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n+1+k}{k}} = \frac{n+1}{2n+1} \quad (5.21)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n+2+k}{k}} = \frac{3(n+1)(n+2)}{2(2n+1)(2n+3)} \quad (5.22)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{1}{\binom{n-1+k}{k}} = 3, \quad n \geq 1 \quad (5.23)$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n-1+k}{k}} = -\frac{5n+2}{n}, \quad n \geq 1 \quad (5.24)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{\binom{\lfloor \frac{n-2}{2} \rfloor + k}{k}} = 3 - \left(8 + \frac{4}{n-1}\right) \frac{1 - (-1)^n}{2}, \quad n \geq 2 \quad (5.25)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{\binom{\lfloor \frac{n-1}{2} \rfloor + k}{k}} = 2(-1)^n + 1, \quad n \geq 1 \quad (5.26)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{\binom{\lfloor \frac{n}{2} \rfloor + k}{k}} = (-1)^n \quad (5.27)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{\binom{\lfloor \frac{n+1}{2} \rfloor + k}{k}} = \frac{(-1)^n + 1}{2} \quad (5.28)$$

5.3.2 Application of Equation (5.21)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2k}{n+k} \frac{3^{2n-k}}{n+k+1} = \binom{2n}{n} \quad (5.29)$$

5.3.3 Application of Equation (5.20)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \frac{2k+1}{\binom{n+k}{k}(n+k+1)} = 1 \quad (5.30)$$

6 Complex Roots of Unity in Series

Remark 6.1 Throughout this chapter, we let $i \equiv \sqrt{-1}$. We assume, unless otherwise specified, that n and k are nonnegative integers, while x , y and z denote real or complex numbers. We also let $[x]$ denote the floor of x (x real).

6.1 Definition of w_n and Basic Orthogonality Relations

6.1.1 Definition of n^{th} Roots of Unity

Let

$$w_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}. \quad (6.1)$$

The n^{th} roots of unity are

$$w_n^k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}, \quad k = 0, 1, 2, \dots, n-1. \quad (6.2)$$

6.1.2 Orthogonality Relations

$$\frac{1}{n} \sum_{k=0}^{n-1} w_n^{kr} = \begin{cases} 1, & r = \alpha n \\ 0, & r \neq \alpha n, \end{cases} \quad n \geq 2, \quad r \text{ and } \alpha \text{ integers} \quad (6.3)$$

$$\sum_{k=0}^{n-1} (-1)^k w_n^{rk} = \begin{cases} 0, & \text{if } n \text{ is even, } n \geq 1 \\ \frac{2}{1+w_n^r} & \text{if } n \text{ is odd} \end{cases} \quad r \text{ an integer} \quad (6.4)$$

Applications of Equation (6.3)

$$\sum_{k=0}^{\infty} \frac{x^{kn}}{(kn)!} = \frac{1}{n} \sum_{k=0}^{n-1} e^{xw_n^k}, \quad n \geq 1 \quad (6.5)$$

Remark 6.2 Let z be a complex number. We let \bar{z} denote the conjugate of z .

$$\frac{1}{n} \sum_{k=0}^{n-1} w_n^{k\alpha} \bar{w}_n^{k\beta} = \begin{pmatrix} 0 \\ \alpha - \beta \end{pmatrix}, \quad n \geq 2, \quad 0 \leq \alpha, \beta \leq n-1 \quad (6.6)$$

α and β are nonnegative integers

Let $f_n(x) = \sum_{j=1}^{n-1} a_j x^j$. Then,

$$f_n(x) = \frac{1}{n} \sum_{j=1}^{n-1} \sum_{k=0}^{n-1} \bar{w}_n^{kj} f_n(xw_n^k), \quad n \geq 2 \quad (6.7)$$

6.2 Complex Roots of Unity in Evaluation of Series

6.2.1 Evaluation of $\sum_{k=0}^n f(kr)$

Remark 6.3 In this section, we assume r is a positive integer. We also assume f is a real or complex valued function whose domain contains the set of nonnegative integers.

$$\sum_{k=0}^n f(kr) = \frac{1}{r} \sum_{k=0}^{rn} \sum_{j=1}^r w_r^{jk} f(k) \quad (6.8)$$

$$\sum_{k=0}^n f(kr) = \frac{1}{r} \sum_{k=0}^{rn} \sum_{j=1}^r \cos \frac{2\pi jk}{r} f(k), \quad f \text{ real valued} \quad (6.9)$$

$$\sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \binom{n}{rk} f(kr) = \frac{1}{r} \sum_{k=0}^n \sum_{j=1}^r \binom{n}{k} \cos \frac{2\pi jk}{r} f(k), \quad f \text{ real valued} \quad (6.10)$$

6.2.2 Applications of Equation (6.8)

$$\sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \binom{n}{rk} = \frac{1}{r} \sum_{k=1}^r (1 + w_r^k)^n \quad (6.11)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = \begin{cases} 2^{n-1}, & n \geq 1 \\ 1, & n = 0 \end{cases} \quad (6.12)$$

$$\sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \binom{n}{rk} = \frac{2^n}{r} \sum_{j=1}^r \left(\cos \frac{\pi j}{r} \right)^n \cos \frac{n\pi j}{r} \quad (6.13)$$

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} = \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right) \quad (6.14)$$

$$\sum_{k=0}^n \binom{3n}{3k} = \frac{1}{3} (2^{3n} + 2(-1)^n) \quad (6.15)$$

$$\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4k} = \frac{1}{4} \left(2^n + 2^{\frac{n}{2}+1} \cos \frac{n\pi}{4} \right) \quad (6.16)$$

$$\sum_{k=0}^n \binom{4n}{4k} = \frac{1}{4} (2^{4n} + (-1)^n 2^{2n+1}) \quad (6.17)$$

$$\sum_{k=1}^n (-1)^k \left(\cos \frac{\pi k}{n} \right)^n = \frac{n}{2^{n-1}}, \quad n \geq 1 \quad (6.18)$$

$$\sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \binom{n}{rk} x^{rk} = \frac{1}{r} \sum_{k=1}^r (1 + xw_r^k)^n \quad (6.19)$$

$$\sum_{k=0}^{\lfloor \frac{n-a}{r} \rfloor} \binom{n}{a+kr} x^{a+kr} = \frac{1}{r} \sum_{k=1}^r (w_r^k)^{-a} (1 + xw_r^k)^n, \quad 0 \leq a \leq n, \quad a \leq r-1, \quad a \text{ an integer} \quad (6.20)$$

$$\sum_{k=0}^{\lfloor \frac{n-a}{r} \rfloor} \binom{n}{a+kr} = \frac{1}{r} \sum_{k=1}^r \left(2 \cos \frac{\pi k}{r} \right)^n \cos \frac{(n-2a)k\pi}{r}, \quad (6.21)$$

$0 \leq a \leq n, \quad a \leq r-1, \quad a \text{ an integer}$

$$\sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n}{3k+1} = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-2)\pi}{3} \right), \quad n \geq 1 \quad (6.22)$$

$$\sum_{k=0}^{\lfloor \frac{2n}{3} \rfloor} \binom{n}{3k-n} = \frac{2^n + 2(-1)^n}{3} \quad (6.23)$$

Remark 6.4 The following identity is W. J. Taylor's Problem 4152 Page 163 of *The American Math. Monthly*, 1945.

$$\frac{1}{2n} \sum_{k=1}^{2n} \left(2 \cos \frac{\pi k}{2n} \right)^{2n} \cos \frac{\alpha k \pi}{n} = \binom{2n}{n-\alpha}, \quad n \text{ and } \alpha \text{ integers, } n \geq 1, -n < \alpha < n \quad (6.24)$$

$$\frac{2^{2n}}{2n} \sum_{k=1}^{2n} \left(\cos \frac{k\pi}{2n} \right)^{2n} = \binom{2n}{n}, \quad n \geq 1 \quad (6.25)$$

$$\sum_{k=1}^n \left(\cos \frac{k\pi}{n} \right)^{2n} = \frac{n}{2^{2n}} \left(\binom{2n}{n} + 2 \right), \quad n \geq 1 \quad (6.26)$$

$$\sum_{k=1}^n \left(\cos \frac{(2k-1)\pi}{2n} \right)^{2n} = \frac{n}{2^{2n}} \left(\binom{2n}{n} - 2 \right), \quad n \geq 1 \quad (6.27)$$

6.2.3 Convolution Formula via Equation (6.8)

Remark 6.5 In this section, we assume g is a real or complex valued function whose domain contains the set of nonnegative integers. We will also assume r is a nonnegative integer.

$$\sum_{k=0}^{\infty} x^{rk} f(k) \sum_{j=0}^{\infty} x^j g(j) = \sum_{j=0}^{\infty} \sum_{k=0}^{\lfloor \frac{j}{r} \rfloor} f(k) g(j-rk) \quad (6.28)$$

$$e^x \left(e^{\frac{x^n}{n}} - 1 \right) = \sum_{j=1}^{\infty} x^j \sum_{k=1}^{\lfloor \frac{j}{n} \rfloor} \frac{1}{n^k k! (j-kn)!}, \quad n \geq 2 \quad (6.29)$$

Combinatorial Numbers and Associated Identities:

Table 1: Stirling Numbers

From the seven unpublished manuscripts of H. W. Gould
 Edited and Compiled by Jocelyn Quaintance

May 3, 2010

1 Notational Conventions for Table 1

Throughout this table, we assume n and k are nonnegative integers. We let B_n denote the n^{th} Bell number. This is a fairly standard notation for the Bell numbers. However, there are many notations for Stirling numbers of the first and second kinds. The following table lists equivalent notations for Stirling numbers of the second kind.

Notation	Author	Source
$S(n, k)$	John Riordan	<i>Combinatorial Identities</i> , 1968
$S(n, k)$	L. Carlitz	numerous papers, mainly that of 1971
$S_2(k, n - k)$	H. W. Gould	various papers circa 1956
$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	Donald Knuth	<i>Art of Computer Programming</i>
S_k^n	G. Pólya	<i>Notes on Combinatorics</i> , 1978
$[C_{n-k}^w(k)]$	J. G. Hagen	<i>Combinationen mit Wiederholungen</i> , 1891
\mathfrak{C}_{k+1}^{n-k}	Niels Nielsen	1906
\mathfrak{S}_n^k	Charles Jordan	1939
$S_n^{(k)}$	Karl Goldberg and Tomlinson Fort	Bureau of Standards, 1959
$\frac{1}{k!} \Delta^k 0^n$	Differences of zero	actuarial work
$\prod_{i=0}^{n-k} \binom{k}{i}$	Goldberg, Leighton, Newman, Zuckerman	1976
$\frac{1}{k!} B_{k,k}^n$	H. W. Gould	private notation of the seven notebooks

Table 1: Equivalent notations for $S(n, k)$, a Stirling number of the second kind

As implied by the last line of Table 1, all of the identities in this volume will use Gould's original notation $B_{k,k}^n$. The reader is urged to remember that $B_{k,k}^n = k!S(n, k)$.

The next table lists equivalent notations for Stirling numbers of the first kind.

Notation	Author	Source
$s(n, k)$	John Riordan	<i>Combinatorial Identities</i> , 1968
$S_1(n, k)$	L. Carlitz	numerous papers, mainly that of 1971
$(-1)^{n-k} S_1(n-1, n-k)$	H. W. Gould	various papers circa 1956
$(-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$	Donald Knuth	<i>Art of Computer Programming</i>
$(-1)^{n-k} S_k^n$	G. Pólya	<i>Notes on Combinatorics</i> , 1978
$(-1)^{n-k} [C_{n-k}(n-1)]$	J. G. Hagen	<i>Combinationen ohne Wiederholungen</i> , 1891
$(-1)^{n-k} C_n^{n-k}$	Niels Nielsen	1906
S_n^k	Charles Jordan	1939
$S_n^{(k)}$	Karl Goldberg	Bureau of Standards, 1959
$(-1)^{n-k} I_n^{(k)}$	Goldberg, Leighton, Newman, Zuckerman	1976
$n!C_k^n$	H. W. Gould	private notation of the seven notebooks

Table 2: Equivalent notations for $s(n, k)$, a Stirling number of the second kind

As implied by the last line of Table 2, all of the identities in this volume will use Gould's original notation C_k^n . The reader is urged to remember that $C_k^n = \frac{s(n,k)}{n!}$.

2 Stirling Numbers of the Second Kind $B_{k,k}^n = k!S(n, k)$

Remark 2.1 Throughout this chapter, we assume r and j are nonnegative integers. We assume x , y , and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x .

2.1 Basis Definition for $B_{k,k}^n$

$$x^n = \sum_{k=0}^n \binom{x}{k} B_{k,k}^n \quad (2.1)$$

2.1.1 Applications of Equation (2.1)

$$\sum_{k=1}^n \frac{(-1)^{k-1}}{k} B_{k,k}^n = \begin{cases} 0, & n \geq 2 \\ 1, & n = 1 \end{cases} \quad (2.2)$$

$$\sum_{k=2}^n \frac{(-1)^k}{k} B_{k,k}^n \sum_{j=1}^{k-1} \frac{1}{j} = \begin{cases} 0, & n \geq 3 \\ 1, & n = 2 \end{cases} \quad (2.3)$$

$$\sum_{k=0}^n (-1)^k B_{k,k}^n = (-1)^n \quad (2.4)$$

$$\sum_{k=0}^n (-1)^k \binom{x+k-1}{k} B_{k,k}^n = (-1)^n x^n \quad (2.5)$$

$$\sum_{k=0}^n (-1)^k k B_{k,k}^n = (-1)^n (2^n - 1) \quad (2.6)$$

$$\sum_{k=0}^n (-1)^k \binom{2k}{k} \frac{1}{2^{2k}} B_{k,k}^n = \frac{(-1)^n}{2^n} \quad (2.7)$$

$$B_{j,j}^n = \sum_{k=j}^n (-1)^{n-k} \binom{k-1}{j-1} B_{k,k}^n \quad (2.8)$$

2.2 Explicit Formulas for $B_{k,k}^n$

$$B_{k,k}^n = (-1)^k \sum_{j=0}^k (-1)^j \binom{k}{j} j^n, \quad 0^0 = 1 \quad (2.9)$$

$$B_{k,k}^n = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n \quad (2.10)$$

2.3 Applications of Equation (2.9)

2.3.1 Evaluations of $\Delta_h^n x^r$

$$\Delta_h^n x^r \equiv \frac{(-1)^n}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} (x+kh)^r = \sum_{k=0}^r \binom{r}{k} x^{r-k} h^{k-n} B_{n,n}^k \quad (2.11)$$

$$\Delta_1^n x^r|_{x=0} = B_{n,n}^r \quad (2.12)$$

2.3.2 Evaluation of $\sum_{k=0}^n (-1)^k \binom{x}{k} k^r$

$$\sum_{k=0}^n (-1)^k \binom{x}{k} k^r = (-1)^n \sum_{k=0}^r \binom{x-k-1}{n-k} \binom{x}{k} B_{k,k}^r, \quad r \geq 1, \quad n \geq 1 \quad (2.13)$$

Applications of Equation (2.13)

$$\sum_{k=0}^n (-1)^k \binom{x}{k} x^r = -x \sum_{k=0}^{r-1} \binom{r-1}{k} \sum_{j=0}^{n-1} (-1)^j \binom{x-1}{j} j^k, \quad r \geq 1, \quad n \geq 1 \quad (2.14)$$

$$\sum_{k=0}^n (-1)^k \binom{x}{k} k^2 = (-1)^n x \binom{x-2}{n-1} + (-1)^n x(x-1) \binom{x-3}{n-2}, \quad n \geq 2 \quad (2.15)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{x}{k} k^3 &= (-1)^n x \binom{x-2}{n-1} + 3(-1)^n x(x-1) \binom{x-3}{n-2} \\ &\quad + (-1)^n x(x-1)(x-2) \binom{x-4}{n-3}, \quad n \geq 3 \end{aligned} \quad (2.16)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{x}{k} k^4 &= (-1)^n x \binom{x-2}{n-1} + 7(-1)^n x(x-1) \binom{x-3}{n-2} \\ &\quad + 6(-1)^n x(x-1)(x-2) \binom{x-4}{n-3} + (-1)^n x(x-1)(x-2)(x-3) \binom{x-5}{n-4}, \quad n \geq 4 \end{aligned} \quad (2.17)$$

$$\sum_{k=0}^n \binom{x+k}{k} k^r = \sum_{k=0}^r \binom{x+k}{k} \binom{x+n+1}{n-k} B_{k,k}^r \quad (2.18)$$

$$\sum_{k=0}^n \binom{x+k}{k} = \binom{n+x+1}{n} \quad (2.19)$$

$$\sum_{k=0}^n \binom{n+k}{k} k^r = \sum_{k=0}^r \binom{n+k}{k} \binom{2n+1}{n-k} B_{k,k}^r \quad (2.20)$$

$$\sum_{k=0}^n \binom{n+k}{k} = \binom{2n+1}{n} \quad (2.21)$$

$$\sum_{k=0}^n \binom{n+k}{k} k = (n+1) \binom{2n+1}{n-1}, \quad n \geq 1 \quad (2.22)$$

$$\sum_{k=0}^n \binom{n+k}{k} k^2 = (n+1) \binom{2n+1}{n-1} + (n+2)(n+1) \binom{2n+1}{n-2}, \quad n \geq 2 \quad (2.23)$$

$$\begin{aligned} \sum_{k=0}^n \binom{n+k}{k} k^3 &= (n+1) \binom{2n+1}{n-1} + 3(n+2)(n+1) \binom{2n+1}{n-2} \\ &\quad + (n+3)(n+2)(n+1) \binom{2n+1}{n-3}, \quad n \geq 3 \end{aligned} \quad (2.24)$$

$$\sum_{k=0}^n k \frac{\binom{n}{k}}{\binom{2n-1}{k}} = \frac{2n}{n+1} \quad (2.25)$$

$$\sum_{k=0}^n \binom{2k}{k} \frac{k^r}{2^{2k}} = \frac{2n+1}{2^{2n}} \binom{2n}{n} \sum_{k=0}^r \binom{n}{k} \frac{1}{2k+1} B_{k,k}^r \quad (2.26)$$

Remark 2.2 The following identity is Problem 4551, P.482, of the *American Math. Monthly*, 1953.

$$\lim_{n \rightarrow \infty} n^{\alpha-k} \sum_{j=0}^{n-1} (-1)^j j^k \binom{\alpha}{j} = \frac{1}{(k-\alpha)\Gamma(-\alpha)}, \quad (2.27)$$

where α is real number which is not a nonnegative integer, and k is a positive integer.

$$\sum_{k=0}^n (-1)^k \binom{x}{k} (z+yk)^r = \sum_{k=0}^r \binom{r}{k} z^{r-k} y^k \sum_{j=0}^k (-1)^j \binom{x}{j} \binom{n-x}{n-j} B_{j,j}^k \quad (2.28)$$

2.3.3 Evaluation of $\sum_{n=0}^{\infty} \frac{n^r x^n}{n!}$

$$\sum_{n=0}^{\infty} \frac{n^r x^n}{n!} = e^x \sum_{k=1}^r \frac{x^k}{k!} B_{k,k}^r \quad (2.29)$$

Applications of Equation (2.29)

$$\sum_{n=0}^{\infty} \frac{n^{r+1} x^n}{n!} = x \sum_{k=0}^r \binom{r}{k} \sum_{n=0}^{\infty} \frac{n^k x^k}{k!} \quad (2.30)$$

$$\sum_{k=0}^{\infty} \frac{k^r \cos kx}{k!} = e^{\cos x} \left(\cos(\sin x) \sum_{k=0}^r B_{k,k}^r \frac{\cos kx}{k!} - \sin(\sin x) \sum_{k=0}^r B_{k,k}^r \frac{\sin kx}{k!} \right) \quad (2.31)$$

$$\sum_{k=0}^{\infty} \frac{k^r \sin kx}{k!} = e^{\cos x} \left(\cos(\sin x) \sum_{k=0}^r B_{k,k}^r \frac{\sin kx}{k!} + \sin(\sin x) \sum_{k=0}^r B_{k,k}^r \frac{\cos kx}{k!} \right) \quad (2.32)$$

2.4 Grunert's Formula

Grunert's Formula

Let S be a function of x . Let $\frac{d^j S}{dx^j}$ denote the j^{th} derivative of S with respect to x . Then,

$$\left(x \frac{d}{dx} \right)^r S = \sum_{k=0}^r B_{k,k}^r \frac{x^k d^k S}{k! dx^k}. \quad (2.33)$$

2.4.1 Applications of Grunert's Formula

$$\sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^n B_{k,k}^n \frac{x^k}{(1-x)^{k+1}}, \quad |x| < 1 \quad (2.34)$$

$$\sum_{k=0}^n k^r x^k = \sum_{k=0}^r B_{k,k}^r \sum_{j=0}^n \binom{j}{k} x^j \quad (2.35)$$

$$\sum_{k=0}^n k^r = \sum_{k=0}^r \binom{n+1}{k+1} B_{k,k}^r \quad (2.36)$$

$$\sum_{k=0}^n k^r = \sum_{k=0}^r (-1)^{r-k} \binom{n+k}{k+1} B_{k,k}^r, \quad r \geq 1 \quad (2.37)$$

$$\sum_{k=0}^n (-1)^k = \frac{1 + (-1)^n}{2} \quad (2.38)$$

$$\sum_{k=0}^n (-1)^k k = (-1)^n \left[\frac{n+1}{2} \right] = \frac{(2n+1)(-1)^n - 1}{4} \quad (2.39)$$

$$\sum_{k=0}^n (-1)^k k^2 = (-1)^n \cdot \frac{n^2 + n}{2} \quad (2.40)$$

$$\sum_{k=0}^n (-1)^k k^3 = \frac{(4n^3 + 6n^2 - 1)(-1)^n + 1}{8} \quad (2.41)$$

$$\sum_{k=0}^n (-1)^k k^4 = (-1)^n \cdot \frac{n^4 + 2n^3 - n}{2} \quad (2.42)$$

$$\sum_{k=0}^n (-1)^{k+1} k^r = \sum_{k=0}^r \left(\binom{n+1}{k+1} - 2^{r+1} \binom{\lfloor \frac{n}{2} \rfloor + 1}{k+1} \right) B_{k,k}^r \quad (2.43)$$

$$\sum_{k=0}^n \binom{n}{k} k^r x^k = (1+x)^n \sum_{k=0}^r \binom{n}{k} \frac{x^k}{(1+x)^k} B_{k,k}^r, \quad x \neq -1 \quad (2.44)$$

$$\sum_{k=0}^n \binom{n}{k} k^r = 2^n \sum_{k=0}^r \frac{1}{2^k} \binom{n}{k} B_{k,k}^r \quad (2.45)$$

$$\sum_{k=0}^n \binom{n}{k} k^3 = 2^n \left(\frac{n}{2} + \frac{3n(n-1)}{4} + \frac{n(n-1)(n-2)}{8} \right) \quad (2.46)$$

$$\sum_{k=0}^n \binom{n}{k} k^4 = 2^n \left(\frac{n}{2} + \frac{7n(n-1)}{4} + \frac{6n(n-1)(n-2)}{8} + \frac{n(n-1)(n-2)(n-3)}{16} \right) \quad (2.47)$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^r (nx)^{2k+1}}{(2k+1)!} = \sum_{k=0}^r x^k n^k \sin \left(nx + \frac{k\pi}{2} \right) B_{k,k}^r \quad (2.48)$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)^r (nx)^{2k}}{(2k)!} = \sum_{k=0}^r x^k n^k \cos \left(nx + \frac{k\pi}{2} \right) B_{k,k}^r \quad (2.49)$$

2.5 Properties of $B_{k,k}^n$

2.5.1 Recurrence Formulas for $B_{k,k}^n$

$$B_{j,j}^n = (-1)^j \sum_{k=0}^n (-1)^k \binom{n}{k} j^{n-k} B_{j,j}^k \quad (2.50)$$

$$B_{j+1,j+1}^{n+1} = (j+1) \left(B_{j+1,j+1}^n + B_{j,j}^n \right) \quad (2.51)$$

Remark 2.3 For the definition of $\Delta_{j,1}$, we refer the reader to Volume 2, Book 1, Chapter 8, Equation (8.1).

$$(-1)^{j+1} B_{j+1,j+1}^{n+1} = (j+1) \left[(-1)^{j+1} B_{j+1,j+1}^n - (-1)^j B_{j,j}^n \right] = [(j+1)\Delta_{j,1}] (-1)^j B_{j,j}^n \quad (2.52)$$

$$\left[\prod_{k=1}^r (j+k)\Delta_{j,1} \right] (-1)^j B_{j,j}^n = (-1)^{j+r} B_{j+r,j+r}^{n+r}, \quad r \geq 1 \quad (2.53)$$

$$\sum_{j=a+1}^r \frac{(-1)^j}{j} B_{j,j}^n = (-1)^r B_{r,r}^{n-1} - (-1)^a B_{a,a}^{n-1}, \quad n \geq 1, \quad a \geq 0 \quad (2.54)$$

$$B_{j,j}^n = \sum_{k=1}^j \binom{j}{k} k! B_{j-k+1,j-k+1}^{n-k}, \quad j \geq 1, \quad n \neq j \quad (2.55)$$

2.5.2 Convolution Property for $B_{k,k}^n$

$$\sum_{k=0}^n \binom{n}{k} B_{r,r}^k B_{j,j}^{n-k} = B_{r+j,r+j}^n \quad (2.56)$$

Applications of Equation (2.56)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (z-k)^{n+j} = \sum_{k=0}^j \binom{z-n}{k} B_{n+k,n+k}^{n+j} \quad (2.57)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (z-k)^{n+1} = \frac{2z-n}{2} (n+1)! \quad (2.58)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (z-k)^{n+2} = \frac{3n^2 + n + 12z^2 - 12zn}{24} (n+2)! \quad (2.59)$$

2.6 Expansions of $\left(\frac{e^x-1}{x}\right)^n$

Remark 2.4 In this section, we assume $D_x^n f(x)$ is the n^{th} derivative of $f(x)$ with respect to x .

$$\left(\frac{e^x-1}{x}\right)^n = \sum_{k=0}^{\infty} B_{n,n}^{k+n} \frac{x^k}{(k+n)!} \quad (2.60)$$

$$D_x^n \left(\frac{e^x-1}{x}\right)^k \Big|_{x=0} = \frac{n!}{(n+k)!} B_{k,k}^{n+k} \quad (2.61)$$

$$D_x^n \left(\frac{x}{e^x-1}\right)^k \Big|_{x=0} = k \binom{k+n}{n} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{j! \binom{n+j}{j} (k+j)} B_{j,j}^{n+j} \quad (2.62)$$

2.7 Polynomial Expansions from $B_{k,k}^n$

Remark 2.5 Throughout this section, we assume $f(x)$ is a polynomial of degree n .

$$f(x) = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} f(k-j) \quad (2.63)$$

2.7.1 Applications of Equation (2.63)

$$\binom{x+n}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n} \quad (2.64)$$

$$\binom{mx}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk-mj}{n}, \text{ where } m \text{ is a complex number} \quad (2.65)$$

$$\binom{x}{n}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \quad (2.66)$$

$$\sum_{k=0}^n \binom{x}{k}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^k (-1)^k \binom{k}{j} \sum_{p=\lceil \frac{k+r-1}{r} \rceil}^n \binom{k-j}{p}^r \quad (2.67)$$

$$\sum_{k=0}^n \binom{n}{k}^3 = \sum_{k=0}^{3n} \binom{n}{k} \sum_{j=0}^k (-1)^k \binom{k}{j} \sum_{p=\lfloor \frac{k+2}{3} \rfloor}^n \binom{k-j}{p}^3 \quad (2.68)$$

$$\sum_{k=0}^n \binom{n}{k}^2 = \sum_{k=0}^{2n} \binom{n}{k} \sum_{j=0}^k (-1)^k \binom{k}{j} \sum_{p=\lfloor \frac{k+1}{2} \rfloor}^n \binom{k-j}{p}^2 = \binom{2n}{n} \quad (2.69)$$

2.8 Polynomial Series via $B_{k,k}^n$

Remark 2.6 Throughout this section, we assume $f(x)$ is a polynomial of degree n .

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} f(k) = e^x \sum_{k=0}^n \frac{x^k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} f(k-j) \quad (2.70)$$

2.8.1 Applications of Equation (2.70)

$$\sum_{k=0}^{\infty} \binom{k+n}{n} \frac{x^k}{k!} = e^x \sum_{k=0}^n \binom{n}{k} \frac{x^k}{k!} \quad (2.71)$$

Dobinski's Formula

$$\sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!} = \frac{1}{e} \sum_{k=0}^n (-1)^k B_{k,k}^n \quad (2.72)$$

3 Bell Numbers B_n

Remark 3.1 Throughout this chapter, we assume r and j are nonnegative integers. We assume x , y , and z are real (complex) numbers. We also let $\lfloor x \rfloor$ denote the floor of x for any real x .

3.1 Definition of B_n

$$B_n = \sum_{k=0}^n \frac{B_{k,k}^n}{k!} \quad (3.1)$$

Stirling's Formula

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad n \geq 1 \quad (3.2)$$

3.2 Properties of Bell Numbers

3.2.1 Bell Number Recurrences

$$B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k, \quad B_0 \equiv 1 \quad (3.3)$$

$$B_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_{k+1} \quad (3.4)$$

3.2.2 Exponential Generating Function

$$e^{e^x - 1} = \sum_{k=0}^{\infty} \frac{x^k}{k!} B_k \quad (3.5)$$

3.3 Schläfli's Modified Bell Number Recurrence

Remark 3.2 *The identities of this section are found in L. Schläfli's "On a generalization given by Laplace of Lagrange's Theorem", Quarterly Journal of Pure and Applied Mathematics, Vol. 2 (1858), pp. 24-31. The reader is also referred to Oliver A. Gross's, "Preferential Arrangements", American Math. Monthly, Vol. 69 (1962), pp. 4-8.*

Schläfli's Recurrence

$$A_n = \sum_{k=0}^{n-1} \binom{n}{k} A_k, \quad n \geq 1, \quad A_0 \equiv 1 \quad (3.6)$$

3.3.1 Alternative Forms of Equation (3.6)

$$2A_n = \sum_{k=0}^n \binom{n}{k} A_k, \quad n \geq 1 \quad (3.7)$$

$$A_n = \sum_{k=0}^n B_{k,k}^n \quad (3.8)$$

3.3.2 Exponential Generating Function

Let $A_n(x) = \sum_{k=0}^n B_{k,k}^n x^k$. Then,

$$\frac{1}{1 - x(e^t - 1)} = \sum_{k=0}^{\infty} A_k(x) \frac{t^k}{k!}. \quad (3.9)$$

$$\frac{1}{2 - e^t} = \sum_{k=0}^{\infty} A_n \frac{t^n}{n!} \quad (3.10)$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^n \binom{k}{r} x^{k-r} B_{k,k}^n = \frac{(e^t - 1)^r}{(1 - (e^t - 1)x)^{r+1}} \quad (3.11)$$

3.4 Dobinski Numbers D_n

3.4.1 Definition of D_n

$$D_n = \sum_{k=0}^n (-1)^k \frac{B_{k,k}^n}{k!} \quad (3.12)$$

Dobinski's Formula

$$D_n = e \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!} \quad (3.13)$$

3.4.2 Properties of Dobinski Numbers

Recurrence Relation

$$(-1)^{n+1} D_n = \sum_{k=0}^n (-1)^k \binom{n}{k} D_{k+1}, \quad D_0 \equiv 1 \quad (3.14)$$

Exponential Generating Function

$$\frac{1}{e^{e^x} - 1} = \sum_{k=0}^{\infty} \frac{x^k}{k!} D_k \quad (3.15)$$

3.5 Functional Bell Number Recurrence

Remark 3.3 In this section, we assume f is a real or complex valued function over the nonnegative integers. We also assume p is a positive integer.

$$\sum_{k=1}^p \binom{n+1}{k} k f(k-1) = (n+1) \sum_{k=0}^p \binom{n}{k} f(k) - (p+1) \binom{n+1}{p+1} f(p) \quad (3.16)$$

4 Shifted Stirling Numbers of the Second Kind $A_{j,n}$

Remark 4.1 Throughout this chapter, we assume r and j are nonnegative integers. We assume x , y , and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x .

4.1 Definition of $A_{j,n}$

$$x^n = \sum_{j=0}^n \binom{x+j-1}{n} A_{j,n} \quad (4.1)$$

4.2 Explicit Formulas for $A_{j,n}$

$$A_{j,n} = \sum_{k=0}^j (-1)^k \binom{n+1}{k} (j-k)^n \quad (4.2)$$

$$A_{j,n} = \sum_{k=0}^j (-1)^{j+k} \binom{n+1}{j-k} k^n \quad (4.3)$$

4.3 Relationships Between $A_{j,n}$ and $B_{j,j}^n$

$$B_{j,j}^n = \sum_{k=0}^n \binom{k-1}{n-j} A_{k,n}, \quad 0 \leq j \leq n \quad (4.4)$$

$$A_{j,n} = \sum_{r=0}^n (-1)^{j+r} \binom{n}{r} j^{n-r} \sum_{k=0}^r \binom{n-k}{j-k} \binom{n+1}{k} B_{k,k}^r \quad (4.5)$$

$$A_{j,n} = \sum_{k=0}^n (-1)^{k+j} \binom{n-k}{n-j} B_{k,k}^n \quad (4.6)$$

4.3.1 Applications of Equation (4.4)

$$\sum_{j=0}^n A_{j,n} = n! \quad (4.7)$$

Remark 4.2 The following two identities are found in Robert Stalley's "A Generalization of the Geometric Series", *American Math. Monthly*, May 1949, Vol. 56, No. 5, pp. 325-327

$$\sum_{k=1}^{\infty} k^n x^k = \sum_{k=1}^n \frac{x^k}{(1-x)^{n+1}} A_{k,n}, \quad |x| < 1, \quad n \geq 1 \quad (4.8)$$

$$\sum_{k=1}^{\infty} k^n x^k = \sum_{j=1}^n \sum_{k=1}^j (-1)^{k+1} \binom{n+1}{k-1} (j-k+1)^n \frac{x^j}{(1-x)^{n+1}}, \quad |x| < 1, \quad n \geq 1 \quad (4.9)$$

Remark 4.3 The following identity can be found in T. M. Apostol's "On the Lerch Zeta Function", *Pacific Journal of Mathematics*, Vol. 1, No. 2, June 1951, pp. 161-167.

$$\frac{z}{ye^z - 1} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{k+1}}{k!(y-1)^{k+1}} \sum_{j=0}^k y^j A_{j,k}, \quad |y| < 1, \quad |ye^z| < 1 \quad (4.10)$$

4.4 Properties of $A_{j,n}$

4.4.1 Indice Equivalence Property

$$A_{j,n} - A_{n-j+1,n} = \begin{cases} 0, & n \geq 1 \\ (-1)^j, & n = 0, \quad j = 0, 1 \end{cases} \quad (4.11)$$

4.4.2 Recurrence Relation

$$A_{j,n+1} = (n+2-j)A_{j-1,n} + jA_{j,n}, \quad j \geq 1 \quad (4.12)$$

4.4.3 Grunert's Formula

Let S be a real or complex valued function over the set of nonnegative integers. Let $\frac{d^k S}{dx^k}$ denote the k^{th} derivative of S with respect to x . Then,

$$\left(x \frac{d}{dx}\right)^n S = \sum_{j=0}^n A_{j,n} \sum_{k=0}^j \binom{j-1}{n-k} \frac{x^k}{k!} \frac{d^k S}{dx^k} \quad (4.13)$$

4.4.4 Evaluation of $\sum_{k=0}^n k^r$

$$\sum_{k=0}^n k^r = \sum_{j=0}^r \binom{n+j}{r+1} A_{j,r}, \quad n \geq 0, \quad r \geq 1 \quad (4.14)$$

5 Worpitzky/Nielsen Numbers $B_{r,q}^n$

Remark 5.1 Throughout this chapter, we assume $r, q, m,$ and j are nonnegative integers. We assume $x, y,$ and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x .

5.1 Definition of $B_{r,q}^n$

$$B_{r,q}^n = \sum_{k=0}^r (-1)^k \binom{q}{k} (r-k)^n \quad (5.1)$$

5.1.1 Connection to $A_{j,n}$

$$A_{j,n} = B_{j,n+1}^n = \sum_{k=0}^j (-1)^k \binom{n+1}{k} (j-k)^n \quad (5.2)$$

5.2 Properties of $B_{r,q}^n$

5.2.1 Index Shift Property

$$B_{k,m+1}^n + (-1)^{m+n-1} B_{n-k+1,m+1}^n = \begin{cases} 0, & m \geq n \geq 1 \\ (-1)^k \binom{m+1}{k}, & m \geq n = 0 \end{cases} \quad (5.3)$$

5.2.2 Relationships to $B_{j,j}^n$

$$(-1)^{m+n} B_{j,j}^n = \sum_{k=0}^{m+1} \binom{k-1}{m-j} B_{k,m+1}^n, \quad m \geq n \quad (5.4)$$

$$B_{j,j}^n = \sum_{k=0}^{m+1} \binom{m-k}{m-j} B_{k,m+1}^n, \quad m \geq n \quad (5.5)$$

$$B_{k,m+1}^n = (-1)^k \sum_{j=0}^m (-1)^j \binom{m-j}{m-k} B_{j,j}^n, \quad m \geq n \quad (5.6)$$

Applications of Equation (5.4)

Remark 5.2 The following identity is a formula given by N. Nielsen in *Traité élémentaire des nombres de Bernoulli*, Paris, 1924, pp. 26-30.

Nielsen's Formula

$$(-1)^{m+n} x^n = \sum_{k=0}^{m+1} \binom{x+k-1}{m} B_{k,m+1}^n, \quad m \geq n \quad (5.7)$$

Recurrence Relation

$$B_{k,m+1}^{n+1} = (m-k+1)B_{k-1,m}^n + kB_{k,m}^n, \quad m \geq n+1 \geq 1 \quad (5.8)$$

5.2.3 Grunert's Formula

Let S be a real or complex valued function over the set of nonnegative integers. Let $\frac{d^j S}{dx^j}$ denote the j^{th} derivative of S with respect to x . Then,

$$\left(x \frac{d}{dx}\right)^n S = (-1)^{m+n} \sum_{k=0}^{m+1} B_{k,m+1}^n \sum_{j=0}^n \binom{k-1}{m-j} \frac{x^j}{j!} \frac{d^j S}{dx^j}, \quad m \geq n \quad (5.9)$$

5.3 Polynomial Expansions from Nielsen's Formula

Remark 5.3 Throughout this section, we assume $f(x)$ is an arbitrary polynomial of degree $\leq m$, namely, $f(x) = \sum_{k=0}^m a_k x^k$.

Nielsen's Polynomial Expansion

$$f(x) = (-1)^m \sum_{k=0}^m \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} f(j-k) \quad (5.10)$$

5.3.1 Applications of Equation (5.10)

Remark 5.4 Throughout this subsection, we assume p is a positive integer.

$$\binom{x}{r}^p = (-1)^{m+rp} \sum_{k=0}^m \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{k-j+r-1}{r}^p, \quad rp \leq m \quad (5.11)$$

$$\binom{x}{r}^p = \sum_{k=1}^{rp-r+1} \binom{x+k-1}{rp} \sum_{j=0}^{k-1} (-1)^j \binom{rp+1}{j} \binom{k-j+r-1}{r}^p, \quad r, p \geq 1 \quad (5.12)$$

$$\binom{x+n}{n} = (-1)^m \sum_{k=0}^m \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{j-k+n}{n}, \quad n \leq m \quad (5.13)$$

$$\binom{x}{n} = (-1)^n \sum_{k=0}^n \binom{x-n+k-1}{n} \sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{j-k+n}{n} \quad (5.14)$$

$$\binom{px}{n} = (-1)^m \sum_{k=0}^m \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{pj-pk}{n}, \quad n \leq m \quad (5.15)$$

$$\binom{x}{n} = (-1)^n \sum_{k=0}^n \binom{x+k-1}{n} \sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{j-k}{n} \quad (5.16)$$

An n^{th} Difference Application of Equation (5.10)

Assume $f(x)$ is a polynomial of degree $\leq m$. Then,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} f(k) = (-1)^{m+n} \sum_{k=0}^m \binom{k-1}{m-n} \sum_{j=0}^k (-1)^j \binom{m+1}{j} f(j-k). \quad (5.17)$$

Applications of Equation (5.17)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{r}^p = (-1)^{rp} \sum_{k=0}^n \sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{k-j+r-1}{r}^p, \quad rp \leq n \quad (5.18)$$

$$\sum_{k=0}^n \sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{k-j+r-1}{r}^p = \begin{cases} 0, & rp < n \\ \frac{(rp)!}{r!^p}, & rp = n \end{cases} \quad (5.19)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{k} \binom{3n-k}{n}^3 = \binom{3n}{2n} \binom{2n}{n} \quad (5.20)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{3n+1}{k} \binom{3n-k}{n}^3 k = (3n+1) \left(1 - \binom{3n}{2n} \binom{2n}{n} \right) \quad (5.21)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 \frac{1}{\binom{3n}{k}^2} = \frac{\binom{2n}{n}}{\binom{3n}{2n}^2} \quad (5.22)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{3n-k}{n}^2 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{n+k}{n}^2 = \binom{2n}{n} \quad (5.23)$$

5.4 Generalized Nielsen Expansions

Remark 5.5 Throughout this section, we assume $f(x)$ is an arbitrary polynomial of degree n , namely, $f(x) = \sum_{k=0}^n a_k x^k$.

Nielsen's First Generalized Polynomial Expansion: Assume $n \leq m$. Then,

$$f(x+y) = (-1)^m \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} f(j-k+y). \quad (5.24)$$

Nielsen's Second Generalized Polynomial Expansion

$$f(x+y) = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^k \binom{k}{j} f(k-j+y) \quad (5.25)$$

5.4.1 Applications of Equation (5.24) and (5.25)

Remark 5.6 In this subsection, we assume p is a positive integer.

$$f(x) = (-1)^m \sum_{k=0}^m (-1)^k \binom{m+1}{k} f(j-m+x-1), \quad n \leq m \quad (5.26)$$

$$\binom{x+y}{r}^p = (-1)^m \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{j-k+y}{r}^p \quad (5.27)$$

$$\begin{aligned} \binom{x+y}{r}^p &= (-1)^{rp+m} \sum_{k=0}^{m+1} \binom{x+k-1}{m} \\ &\quad \cdot \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{k-j-y+r-1}{r}^p, \quad rp \leq m \end{aligned} \quad (5.28)$$

$$\binom{x+y}{r}^p = \sum_{k=0}^{rp+1} \binom{x+k-1}{rp} \sum_{j=0}^k (-1)^j \binom{rp+1}{j} \binom{k-j-y+r-1}{r}^p \quad (5.29)$$

$$\binom{x-y}{r}^p = \sum_{k=0}^{rp+1} \binom{x+k-1}{rp} \sum_{j=0}^k (-1)^j \binom{rp+1}{j} \binom{k-j+y+r-1}{r}^p \quad (5.30)$$

$$\binom{1-y}{r}^p = \sum_{k=0}^{rp} (-1)^k \binom{rp+1}{k} \binom{rp-k+y+r-1}{r}^p \quad (5.31)$$

$$\sum_{k=0}^{rp} (-1)^k \binom{rp+1}{k} \binom{rp-k+1}{r}^p = 0, \quad r \geq 1 \quad (5.32)$$

$$\sum_{k=0}^{rp-r+1} (-1)^k \binom{rp+1}{k} \binom{rp-k}{r}^p = 1 \quad (5.33)$$

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{k} \binom{2n-k}{n}^2 = 1 \quad (5.34)$$

$$\sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{2n-k}{n}^2 \frac{1}{2n+1-k} = \frac{1}{2n+1} \quad (5.35)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} \frac{1}{2n+1-k} = \frac{1}{(2n+1) \binom{2n}{n}} \quad (5.36)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{3n+1}{k} \binom{3n-k}{n}^2 = 1 \quad (5.37)$$

$$\sum_{k=0}^{3n} (-1)^k \binom{4n+1}{k} \binom{4n-k}{n}^2 = 1 \quad (5.38)$$

$$\binom{px+py}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{pk-pj+py}{n} \quad (5.39)$$

$$\binom{px+py}{n} = (-1)^m \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{pj-pk+py}{n}, \quad n \leq m \quad (5.40)$$

$$\begin{aligned} \binom{px+py}{n} &= (-1)^{m+n} \sum_{k=0}^{m+1} \binom{x+k-1}{m} \\ &\quad \cdot \sum_{j=0}^k (-1)^j \binom{m+1}{j} \binom{pk-pj-py+n-1}{n}, \quad n \leq m \end{aligned} \quad (5.41)$$

$$\binom{2x+2y}{n} = \sum_{k=0}^{n+1} \binom{x+k-1}{n} \sum_{j=0}^k (-1)^j \binom{n+1}{j} \binom{2k-2j-2y+n-1}{n} \quad (5.42)$$

$$\binom{2y+2}{n} = \sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{3n-2k-2y-1}{n} \quad (5.43)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{3n-2k+1}{n} = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases} \quad (5.44)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{3n-2k-1}{n} = \binom{2}{n} \quad (5.45)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{2n-2k+y}{n} = \binom{n-y+1}{n} \quad (5.46)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{2n-2k}{n} = n+1 \quad (5.47)$$

Remark 5.7 The following identity is the solution of B. C. Wong's Problem 3399 of *The American Math. Monthly*, Vol. 36, No. 10, December 1929.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{k} \binom{2n-2k}{n} = n+1 \quad (5.48)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{2n-2k+1}{n} = 1 \quad (5.49)$$

Remark 5.8 The following identity is the solution of B. C. Wong's Problem 3426 of *The American Math. Monthly*, May 1930.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{k} \binom{2n-2k-1}{n} = 1, \quad n \geq 1 \quad (5.50)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{2n-2k-1}{n} = \binom{n+2}{n} \quad (5.51)$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{k} \binom{2n-2k-1}{n} = \begin{cases} 1, & n = 0 \\ \frac{n^2+n}{2}, & n \geq 1 \end{cases} \quad (5.52)$$

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k} \binom{pn-pk+r}{n} = \binom{n-r+p-1}{n} \quad (5.53)$$

$$\sum_{k=0}^{\lfloor \frac{(p-1)n}{p} \rfloor} (-1)^k \binom{n+1}{k} \binom{pn-pk}{n} = \binom{n+p-1}{n} \quad (5.54)$$

$$\sum_{k=0}^{\lfloor \frac{2n}{3} \rfloor} (-1)^k \binom{n+1}{k} \binom{3n-3k}{n} = \binom{n+2}{n} \quad (5.55)$$

5.5 Nielsen Numbers $\beta_p^{m,n}(\alpha)$

Remark 5.9 Throughout this section, we assume p is a nonnegative integer while α is a real or complex number.

5.5.1 Definition of $\beta_p^{m,n}(\alpha)$

$$\beta_p^{m,n}(\alpha) = \sum_{k=0}^p (-1)^k \binom{m+1}{k} (\alpha + p - k)^n \quad (5.56)$$

Remark 5.10 This definition is found on Page 31 of Traité élémentaire des nombres de Bernoulli, by Niels Nielsen, Gauthier-Villars, Paris, 1923.

5.5.2 Relationship to Worpitzky Numbers

$$\beta_p^{m,n}(0) = B_{p,m+1}^n \quad (5.57)$$

$$\beta_p^{m,n}(\alpha) = \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} B_{p,m+1}^k \quad (5.58)$$

5.5.3 Polynomial Expansions via Nielsen Numbers

Remark 5.11 This identity is found on Page 28 of Traité élémentaire des nombres de Bernoulli by Niels Nielsen.

$$(-1)^{m+n} (x - \alpha)^n = \sum_{k=0}^{m+1} \binom{x+k-1}{m} \beta_k^{m,n}(\alpha), \quad m \geq n \quad (5.59)$$

6 Stirling Numbers of the First Kind $C_k^n = \frac{s(n,k)}{n!}$

Remark 6.1 Throughout this chapter, we assume r, q, m , and j are nonnegative integers. We assume x, y , and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x .

6.1 Basis Definition of C_k^n

$$\binom{x}{n} = \sum_{k=0}^n C_k^n x^k \quad (6.1)$$

6.1.1 Derivatives of $\binom{x}{n}$

Remark 6.2 For this subsection, we let D_x^p denote the p^{th} derivative with respect to x .

$$D_x^1 \binom{x}{n} = \binom{x}{n} \sum_{k=1}^n \frac{1}{k+x-n}, \quad n \geq 1 \quad (6.2)$$

$$D_x^1 \binom{x}{n} \Big|_{x=0} = \frac{(-1)^{n-1}}{n} = C_1^n, \quad n \geq 1 \quad (6.3)$$

$$D_x^2 \binom{x}{n} = \binom{x}{n} \left[\left(\sum_{k=1}^n \frac{1}{k+x-n} \right)^2 - \sum_{k=1}^n \frac{1}{(k+x-n)^2} \right], \quad n \geq 1 \quad (6.4)$$

$$D_x^2 \binom{x}{n} \Big|_{x=0} = \frac{2}{n} (-1)^n \sum_{k=1}^{n-1} \frac{1}{k} = 2C_2^n, \quad n \geq 2 \quad (6.5)$$

6.2 Properties of C_k^n

6.2.1 Recurrence Formula

$$(n+1)C_k^{n+1} = C_{k-1}^n - nC_k^n, \quad k \geq 1 \quad (6.6)$$

6.2.2 Convolution Formula

$$\sum_{r=0}^n C_j^r C_k^{m-r} = C_{k+j}^m \binom{k+j}{j} \quad (6.7)$$

6.2.3 Orthogonality Properties

$$\sum_{j=0}^n B_{j,j}^n C_k^j = \begin{pmatrix} 0 \\ n - k \end{pmatrix} \quad (6.8)$$

$$\sum_{j=0}^n C_j^n B_{k,k}^j = \begin{pmatrix} 0 \\ n - k \end{pmatrix} \quad (6.9)$$

Remark 6.3 The following identity is a formula of Frank R. Olson from *The American Math. Monthly*, October 1956, Vol. 63, No. 8, p. 612.

$$\sum_{j=0}^n C_j^r B_{k,k}^{n+j-1} = \begin{cases} 0, & k < r \\ r^{n-1}, & k = r \end{cases} \quad (6.10)$$

6.2.4 Inversion Formulas

Assume b_k is independent of n , and $a_n = \sum_{k=0}^n B_{k,k}^n b_k$. Then, $b_n = \sum_{k=0}^n C_k^n a_k$. (6.11)

Let $r \geq n$. Assume b_k is independent of n , and $a_n = \sum_{k=0}^r B_{n,n}^k b_k$. Then, $b_n = \sum_{k=0}^r C_n^k a_k$. (6.12)

6.2.5 Alternating Sum Formulas

$$\sum_{k=j}^n (-1)^k C_j^k = (-1)^n \sum_{k=j}^n (-1)^{k-j} \binom{k}{j} C_k^n \quad (6.13)$$

$$\sum_{k=1}^n \frac{1}{k} = \sum_{k=1}^n (-1)^{n-k} k C_k^n, \quad k \geq 1 \quad (6.14)$$

6.3 Functional Expansions Involving C_k^n

6.3.1 Expansions of $\prod_{k=0}^{n-1} (1 + kx)$

$$\prod_{k=0}^{n-1} (1 + kx) = n! \sum_{k=0}^n (-1)^k C_{n-k}^n x^k, \quad n \geq 1 \quad (6.15)$$

$$\frac{n!}{\prod_{k=1}^n (1 - kx)} = \sum_{k=0}^{\infty} B_{n,n}^{k+n} x^k, \quad n \geq 1, \quad |x| < \frac{1}{n} \quad (6.16)$$

6.3.2 Expansions of $\binom{x+n}{n}$

$$\binom{x+n}{n} = \sum_{k=0}^n x^k \sum_{j=0}^n \binom{n}{j} C_k^j \equiv \sum_{k=0}^n q_k^n x^k \quad (6.17)$$

$$\text{If } \binom{x+n}{n} = \sum_{k=0}^n q_k^n x^k, \text{ then, } C_k^r = \sum_{j=0}^r (-1)^{j+r} \binom{r}{j} q_k^j. \quad (6.18)$$

6.3.3 Expansions of $\binom{x}{n}$ Involving $B_{j,j}^x$

Remark 6.4 For this subsection, we define, for arbitrary real or complex x , the generalized Stirling number of the second kind $B_{j,j}^x$ as follows.

$$B_{j,j}^x = \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} k^x \quad (6.19)$$

Note that Equation (6.19) implies

$$z^x = \sum_{k=0}^{\infty} B_{k,k}^x \binom{z}{k}. \quad (6.20)$$

$$\binom{x}{n} = \sum_{k=0}^{\infty} B_{k,k}^x (C_n^k + C_n^{k-1}) \quad (6.21)$$

$$x = 1 + \sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} B_{k,k}^x \quad (6.22)$$

$$\sum_{k=0}^n C_k^m \binom{k}{j} = C_j^m + C_j^{m-1} \quad (6.23)$$

$$\sum_{k=1}^n k C_k^n = \frac{(-1)^n}{n(n-1)}, \quad n \geq 2 \quad (6.24)$$

Two Orthogonality Relationships

$$\text{Let } B_{j,j}^x = \sum_{k=0}^{\infty} \binom{x}{k} E_j^k, \text{ where } E_j^k = \sum_{r=0}^j (-1)^{r+j} \binom{j}{r} (r-1)^k. \quad (6.25)$$

Then,

$$\sum_{j=0}^{\infty} (C_n^j + C_n^{j-1}) E_j^k = \binom{0}{n-k}, \quad (6.26)$$

and

$$\sum_{j=0}^{\infty} E_n^j (C_j^k + C_j^{k-1}) = \binom{0}{n-k}. \quad (6.27)$$

6.3.4 Fractional Binomial Sum Expansions

$$\sum_{k=0}^n (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{(-1)^n}{\binom{x+n}{n}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} C_{2k+1}^{n+1} x^{2k} \quad (6.28)$$

$$\sum_{k=0}^1 (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{1}{(x+1)} \quad (6.29)$$

$$\sum_{k=0}^2 (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{x^2 + 2}{(x+2)(x+1)} \quad (6.30)$$

$$\sum_{k=0}^3 (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{6x^2 + 6}{(x+3)(x+2)(x+1)} \quad (6.31)$$

$$\sum_{k=0}^4 (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{x^4 + 35x^2 + 24}{(x+4)(x+3)(x+2)(x+1)} \quad (6.32)$$

6.3.5 Expansion of $(1+z)^{\frac{1}{z}}$

Remark 6.5 The following identity is found in “On the Expansion of $(1+x)^{\frac{1}{x}}$ in Ascending Powers of x ”, by Percival Frost, *Quarterly Journal of Mathematics, London: Vol. 7, No. 28, Feb. 1866.*

$$(1+z)^{\frac{1}{z}} = \sum_{j=0}^{\infty} z^j \sum_{k=0}^{\infty} C_k^{k+j}, \quad \text{where} \quad (6.33)$$

$$C_{k-j}^k = \frac{(-1)^j}{k!} \sum_{r=0}^j \binom{j+k}{j-r} \binom{r-k}{j+r} \frac{(-1)^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} s^{r+j} \quad (6.34)$$

6.4 A Derivative Expansion Involving C_k^n

Remark 6.6 Throughout this subsection, we let $D_x^n f(x)$ denote the n^{th} derivative of $f(x)$ with respect to x .

$$D_x^n f(\ln x) = \frac{n!}{x^n} \sum_{k=0}^n C_k^n D_x^k f(z), \quad \text{where } z = \ln x \quad (6.35)$$

6.4.1 Applications of Equation (6.35)

$$D_z^n (\ln z)^j = \frac{n!j!}{z^n} \sum_{k=0}^j \frac{1}{k!} (\ln z)^k C_{j-k}^n \quad (6.36)$$

$$D_z^n (\ln z)^2 = \frac{2(-1)^{n-1}(n-1)!}{z^n} \left(\ln z - \sum_{k=1}^{n-1} \frac{1}{k} \right), \quad n \geq 2 \quad (6.37)$$

$$\sum_{k=0}^n B_{k,k}^n \frac{z^k}{k!} D_z^k (\ln z)^n = n! \quad (6.38)$$

6.4.2 Inversion of Equation (6.35)

$$C_j^n = \frac{(-1)^j z^n}{j!n!} \sum_{k=0}^j (-1)^k \binom{j}{k} (\ln z)^{j-k} D_z^n (\ln z)^k \quad (6.39)$$

Applications of Equation (6.39)

$$C_j^n = \frac{1}{j!n!} D_z^n (\ln z)^j \Big|_{z=1} \quad (6.40)$$

$$(\ln(z+1))^j = j! \sum_{k=j}^{\infty} C_j^k z^k, \quad |z| < 1 \quad (6.41)$$

$$(z+1)^x = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} C_k^j z^j \quad (6.42)$$

$$(x+1)^x = \sum_{j=0}^{\infty} x^j \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} C_k^{j-k} \quad (6.43)$$

$$x^j = j! \sum_{k=0}^{\infty} (e^x - 1)^k C_j^k, \quad |e^x| < 1 \quad (6.44)$$

6.5 Explicit Formulas for C_k^n using Bernoulli Polynomials and $B_{j,j}^n$

Remark 6.7 Let t be a real (complex) number. We define $B_k^{(n)}(t)$ as the general Bernoulli polynomial of order n and degree k by the generating function relation

$$\sum_{k=0}^{\infty} B_k^{(n)}(t) \frac{x^k}{k!} = \frac{x^n e^{tx}}{(e^x - 1)^n}, \quad |x| < 2\pi. \quad (6.45)$$

An excellent reference for properties of $B_k^{(n)}(t)$ is Calculus of Finite Differences by Charles Jordan, Second Edition, Chelsea Publishing, New York, 1947.

$$C_{n-k}^n = \frac{1}{k!(n-k)!} B_k^{(n+1)}(1) \quad (6.46)$$

$$C_{n-k}^n = \frac{1}{k!(n-k)!} D_x^k \left(e^x \left(\frac{x}{e^x - 1} \right)^{n+1} \right) \Big|_{x=0} \quad (6.47)$$

$$C_{n-k}^m = \frac{1}{k!(n-k)!} \sum_{j=0}^k \binom{k}{j} D_x^j \left(\frac{x}{e^x - 1} \right)^{n+1} \Big|_{x=0} \quad (6.48)$$

$$n!C_{n-k}^m = \binom{n-1}{k} \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \frac{k!}{(k+nj)!} B_{nj,nj}^{k+nj} \quad (6.49)$$

Inverse of Identity (6.49)

$$\frac{1}{n!} B_{n,n}^{k+n} = \binom{k+n}{k} \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \frac{(jn)!}{\binom{jn-1}{k}} C_{jn-k}^{jn} \quad (6.50)$$

$$C_{n-k}^n = \frac{1}{(n-k)!k!} \sum_{j=0}^k \binom{k}{j} \sum_{s=0}^j (-1)^s \binom{j+1}{s+1} \frac{j!}{(j+sn+s)!} B_{sn+s,sn+s}^{j+sn+s} \quad (6.51)$$

$$C_{n-k}^n = \frac{1}{(n-k)!k!} \sum_{j=0}^k \binom{k}{j} \sum_{s=0}^j (-1)^s \frac{j+1}{s+1} \binom{j}{s} \frac{1}{(sn+s)! \binom{j+sn+s}{j}} B_{sn+s,sn+s}^{j+sn+s} \quad (6.52)$$

$$C_{n-k}^n = \frac{1}{(n-k)!k!} \sum_{j=0}^k \binom{k}{j} (n+1) \binom{n+1+j}{j} \cdot \sum_{s=0}^j (-1)^s \binom{j}{s} \frac{1}{j!(n+1+s) \binom{j+s}{j}} B_{s,s}^{j+s} \quad (6.53)$$

$$C_{n-k}^n = \frac{1}{(n-k)!k!} \sum_{j=0}^k \binom{k}{j} \binom{n+j}{j} \sum_{s=0}^j (-1)^s \binom{j}{s} \frac{n+j+1}{n+s+1} \frac{1}{s! \binom{j+s}{s}} B_{s,s}^{j+s} \quad (6.54)$$

6.6 Schläfli's Formulas for C_k^n

Remark 6.8 The identities of this section can be found in the following two papers, both written by L. Schläfli: “Sur les coefficients du développement du produit

$1(1+x)(1+2x)\dots(1+(n-1)x)$ suivant les puissances ascendantes de x ”, *Jour. reine u. angew. Math.*, Volume 43, 1852, pp. 1-22: “Ergänzung der Abhandlung über die Entwicklung des Produkts $1(1+x)(1+2x)\dots(1+(n-1)x) = \prod^n(x)$ ”, *Jour. reine u. angew. Math.*, Volume 67, 1867, pp. 179-182.

$$C_{n-k}^m = \frac{(-1)^k}{n!} \sum_{j=0}^k \binom{k+n}{k-j} \binom{k-n}{k+j} \frac{1}{j!} B_{j,j}^{j+k} \quad (6.55)$$

$$C_{n-k}^n = \frac{1}{n!} \sum_{j=0}^k (-1)^j \binom{n+j-1}{n-k-1} \binom{n+k}{n+j} \frac{1}{j!} B_{j,j}^{j+k} \quad (6.56)$$

6.6.1 Inverse Relations of Equation (6.55)

$$B_{n-k,n-k}^n = (-1)^k (n-k)! \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} (k+j)! C_j^{k+j} \quad (6.57)$$

7 Appendix A: Contour Integral Formulas for Stirling Numbers

Remark 7.1 Throughout this appendix, we assume $i \equiv \sqrt{-1}$. We also let γ denote a simple closed curve around the origin, namely, the unit circle.

7.1 Contour Integrals for $B_{k,k}^n$

$$B_{k,k}^n = \frac{n!}{2\pi i} \int_{\gamma} \frac{(e^z - 1)^k}{z^{n+1}} dz \quad (7.1)$$

$$B_{k,k}^n = \binom{n-1}{k-1} \frac{k!(n-k)!}{2\pi i} \int_{\gamma} \frac{e^z (e^z - 1)^{k-1}}{z^n} dz \quad (7.2)$$

7.2 Contour Integrals for C_k^n

$$C_k^n = \frac{1}{2\pi i} \int_{\gamma} \binom{z}{n} \frac{1}{z^{k+1}} dz \quad (7.3)$$

$$C_k^n = \frac{1}{2\pi} \int_0^{2\pi} \binom{e^{it}}{n} e^{-ikt} dt \quad (7.4)$$

$$k!C_k^n = \frac{1}{2\pi i} \int_{\gamma} \frac{(\ln(z+1))^k}{z^{n+1}} dz \quad (7.5)$$

$$k!C_k^n = \frac{1}{2\pi i} \int_{\gamma} \frac{z^k e^z}{(e^z - 1)^{n+1}} dz \quad (7.6)$$

$$k!C_k^n = \frac{k}{2n\pi i} \int_{\gamma} \frac{z^{k-1}}{(e^z - 1)^n} dz \quad (7.7)$$

$$n!C_{n-k}^n = \binom{n-1}{k} \frac{k!}{2\pi i} \int_{\gamma} \frac{z^{n-k-1}}{(e^z - 1)^n} dz \quad (7.8)$$

7.2.1 Extension of Identity (7.7)

Remark 7.2 In the following identity, assume α is a complex number with $\Re(\alpha) > 1$.

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{z^{\alpha-1}}{(e^z - 1)^n} dz = n \sum_{k=0}^n C_k^n \sum_{j=n}^\infty \frac{1}{j^{\alpha+1-k}} \quad (7.9)$$

8 Appendix B: Asymptotic Approximations for Stirling Numbers

Remark 8.1 Many of the identities in this appendix are found in Charles Jordan's "On Stirling's numbers", *Tohoku Math. Journal*, Vol. 37, 1933, pp. 254-278.

8.1 Approximations Involving $B_{k,k}^n$

$$\lim_{n \rightarrow \infty} \frac{B_{k,k}^{n+1}}{B_{k,k}^n} = k \quad (8.1)$$

$$\lim_{n \rightarrow \infty} \frac{B_{n,n}^{n+k}}{(n+k)^{2k}} = \frac{1}{k!2^k} \quad (8.2)$$

8.2 Approximations Involving C_k^n

$$|C_n^{n+k}| < \frac{e^{k+n}}{n!}, \quad n \geq 1, \quad k \geq 0 \quad (8.3)$$

$$\lim_{n \rightarrow \infty} \frac{n! |C_{n-k}^n|}{n^{2k}} = \frac{1}{k!2^k} \quad (8.4)$$

$$\lim_{n \rightarrow \infty} \left| \frac{C_{k+1}^n}{(\ln n + \gamma) C_k^n} \right| = \frac{1}{k}, \quad \text{where } \gamma \text{ is Euler's constant} \quad (8.5)$$

Remark 8.2 The following identity, due to H. W. Becker, is found in the *American Math. Monthly*, Vol. 50, 1943, Page 327.

$$\lim_{r \rightarrow \infty} \frac{\Delta_{n,1}^r |(n+1)! C_k^{n+1}|}{|(n+r+1)! C_k^{n+r+1}|} = \frac{1}{e} \quad (8.6)$$

9 Appendix C: Number Theoretic Definitions of Stirling Numbers

9.1 Kramp-Ettinghausen Definitions of Stirling Numbers

$$S_1(n-1, k) = \sum \frac{n(n-1)(n-2)\dots(n-\gamma+1)}{\alpha_1!\alpha_2!\alpha_3!\dots 2^{\alpha_1}3^{\alpha_2}4^{\alpha_3}}, \quad (9.1)$$

where the summation is over all possible integers $\alpha_\beta \geq 1$ such that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = k$, and $\gamma = \alpha_1 + \alpha_2 + \alpha_3 + \dots + k$.

$$S_2(n, k) = \sum \frac{(n+k)(n+k-1)\dots(n+k-\gamma+1)}{\alpha_1!\alpha_2!\alpha_3!\dots (2!)^{\alpha_1}(3!)^{\alpha_2}(4!)^{\alpha_3}}, \quad (9.2)$$

where the summation is over all possible integers $\alpha_\beta \geq 1$ such that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = k$, and $\gamma = \alpha_1 + \alpha_2 + \alpha_3 + \dots + k$.

9.2 Iterated Definitions of $S_1(n, k)$ and $S_2(n, k)$

$$S_1(n, k) = \sum_{j_k=1}^{n-k+1} j_k \sum_{j_{k-1}=j_k+1}^{n-k+2} j_{k-1} \sum_{j_{k-2}=j_{k-1}+1}^{n-k+3} j_{k-2} \dots \sum_{j_2=j_3+1}^{n-1} j_2 \sum_{j_1=j_2+1}^n j_1 \quad (9.3)$$

$$S_1(n, k) = \prod_{i=1}^k \sum_{j_i=j_{i+1}+1}^{n-i+1} j_i, \quad \text{with } j_{k+1} \equiv 0 \quad (9.4)$$

$$S_2(n, k) = \sum_{j_k=1}^n j_k \sum_{j_{k-1}=j_k}^n j_{k-1} \sum_{j_{k-2}=j_{k-1}}^n j_{k-2} \dots \sum_{j_2=j_3}^n j_2 \sum_{j_1=j_2}^n j_1 \quad (9.5)$$

$$S_2(n, k) = \prod_{i=1}^k \sum_{j_i=j_{i+1}}^n j_i, \quad \text{with } j_{k+1} \equiv 1 \quad (9.6)$$

9.2.1 Restatements of Equations (9.3) and (9.5)

$$S_1(n, k) = \sum_{1 \leq r_1 < r_2 < r_3 < \dots < r_k \leq n} r_1 r_2 r_3 \dots r_k, \quad \text{where } r_i \text{ is a positive integer} \quad (9.7)$$

$$S_2(n, k) = \sum_{1 \leq r_1 \leq r_2 \leq r_3 \leq \dots \leq r_k \leq n} r_1 r_2 r_3 \dots r_k, \quad \text{where } r_i \text{ is a positive integer} \quad (9.8)$$

9.2.2 Application of Equation (9.5)

$$S_1(n, n-k) = n! \prod_{i=1}^k \sum_{j_i=i}^{j_{i+1}-1} \frac{1}{j_i}, \quad \text{with } j_{k+1} \equiv n+1 \quad (9.9)$$

9.3 Properties of $S_1(n, k)$ and $S_2(n, k)$

$$S_1(n-1, k) = (-1)^k n! C_{n-k}^n \quad (9.10)$$

$$S_2(n, k) = \frac{1}{n!} B_{n,n}^{n+k} \quad (9.11)$$

$$S_1(n, k) = S_1(n-1, k) + nS_1(n-1, k-1), \quad n \geq 1, k \geq 1 \quad (9.12)$$

$$S_2(n, k) = S_2(n-1, k) + nS_2(n, k-1), \quad n \geq 1, k \geq 1 \quad (9.13)$$

$$S_1(n-1, k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} S_2(j, k) \quad (9.14)$$

$$S_2(n, k) = S_1(-n-1, k) \quad (9.15)$$

$$S_1(n, k) = S_2(-n-1, k) \quad (9.16)$$

9.3.1 Generalization of $S_2(n, k)$

Remark 9.1 In the following identity, we assume z is an arbitrary real or complex number.

$$S_2(z, k) = \sum_{j=0}^k \binom{k+z}{k+j} \binom{k-z}{k-j} S_2(j, k) \quad (9.17)$$

9.3.2 Basis Expansions Involving $S_1(n, k)$ and $S_2(n, k)$

Remark 9.2 In the following two formulas, we assume x is a real or complex number.

$$\prod_{k=0}^{n-1} (1+kx) = \sum_{k=0}^n S_1(n-1, k) x^k, \quad n \geq 1 \quad (9.18)$$

$$\frac{1}{\prod_{k=1}^n (1-kx)} = \sum_{k=0}^{\infty} S_2(n, k) x^k, \quad n \geq 1, |x| < \frac{1}{n} \quad (9.19)$$

9.3.3 Hagen Recurrences

Remark 9.3 *The following identities are found in Hagen's Synopsis der hoeheren Mathematik, Berlin 1891, Volume I, Page 60.*

$$S_1(n-1, k) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{n-j}{k+1-j} S_1(n-1, j), \quad n \geq 1 \quad k \geq 1 \quad (9.20)$$

$$nC_{n-k}^n = (-1)^k \sum_{j=0}^k (-1)^j \binom{n-j}{k+1-j} C_{n-j}^n \quad (9.21)$$

$$S_2(n, k) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{-n-j}{k+1-j} S_2(n, j), \quad k \geq 1 \quad (9.22)$$

$$-nB_{n,n}^{n+k} = \sum_{j=0}^k \binom{-n-j}{k+1-j} B_{n,n}^{n+j} \quad (9.23)$$

Table for Combinatorial Numbers and Associated Identities: Table 2

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

May 3, 2010

1 Bernoulli Numbers \mathcal{B}_n

Remark 1.1 Throughout this chapter, we assume n and p are nonnegative integers. We assume x is a real or complex number. Furthermore, for any real x , we let $[x]$ denote the floor of x .

1.1 Generating Function Definition of \mathcal{B}_n

$$\sum_{k=0}^{\infty} \mathcal{B}_k \frac{x^k}{k!} = \frac{x}{e^x - 1}, \quad |x| < 2\pi \quad (1.1)$$

1.2 Alternative Definition for \mathcal{B}_n

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k} n^{p+1-k} \mathcal{B}_k, \quad n \geq 1 \quad (1.2)$$

1.3 Explicit Formulas of \mathcal{B}_n

$$\mathcal{B}_p = \sum_{j=0}^p \frac{1}{j+1} \sum_{k=0}^j (-1)^k \binom{j}{k} k^p \quad (1.3)$$

$$\mathcal{B}_n = \frac{2}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k+1} B_{k,k}^{n+1} \sum_{j=1}^k \frac{1}{j}, \quad \text{where } B_{k,k}^n = \sum_{j=0}^k (-1)^k \binom{k}{j} (k-j)^n \quad (1.4)$$

$$\mathcal{B}_n = n! \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \frac{B_{k,k}^{n+k}}{(n+k)!} \quad (1.5)$$

1.3.1 Alternative Formulations of Equation (1.3)

$$\mathcal{B}_n = \sum_{k=0}^n \frac{(-1)^k}{k+1} B_{k,k}^n \quad (1.6)$$

$$\mathcal{B}_n = \sum_{k=0}^n \frac{(-1)^k}{k+1} \sum_{j=0}^n \binom{j-1}{n-k} A_{j,n}, \quad \text{where } A_{j,n} = \sum_{k=0}^j (-1)^k \binom{n+1}{k} (j-k)^n \quad (1.7)$$

$$\mathcal{B}_n = (-1)^n \sum_{k=1}^n A_{k,n} \frac{(-1)^{k-1}}{(n+1) \binom{n}{k-1}}, \quad n \geq 1 \quad (1.8)$$

$$\mathcal{B}_n = \sum_{k=1}^n (-1)^k \frac{A_{k,n}}{(n+1) \binom{n}{k}}, \quad n \geq 1 \quad (1.9)$$

1.4 Vandiver's Formulas for \mathcal{B}_n

Remark 1.2 *The formulas in this section are the work of H. S. Vandiver. The pertinent papers are "On generalizations of the numbers of Bernoulli and Euler", Proc. of the National Academy of Sciences, Vol. 23, 1937, pp. 555-559 (also see Proc. of the National Academy of Sciences, Vol. 25, 1939, pp. 197-201), and "Explicit expressions for generalized Bernoulli numbers", Duke Math. Journal, Vol. 8, No. 3, Sept. 1941, pp. 575-584.*

$$\mathcal{B}_n = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \frac{1}{k+1} \sum_{j=0}^k j^n \quad (1.10)$$

$$\mathcal{B}_n = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \frac{1}{k+1} \sum_{j=0}^{k+1} j^n, \quad n \geq 1 \quad (1.11)$$

$$\mathcal{B}_n = (-1)^n \frac{n!}{n+1} + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{k+1} \sum_{j=0}^k j^n, \quad n \geq 1 \quad (1.12)$$

1.5 Properties of \mathcal{B}_n

1.5.1 Recursive Relation

$$\sum_{k=0}^n \binom{n}{k} \mathcal{B}_k = (-1)^n \mathcal{B}_n \quad (1.13)$$

1.5.2 Parity Properties

$$\mathcal{B}_{2n} = \sum_{k=0}^{2n} \frac{(-1)^k}{k+1} B_{k,k}^{2n} \quad (1.14)$$

$$\mathcal{B}_{2n+1} = \sum_{k=0}^{2n+1} \frac{(-1)^k}{k+1} B_{k,k}^{2n+1} = 0, \quad n \geq 1 \quad (1.15)$$

$$\mathcal{B}_{2n} = \sum_{k=0}^{2n} (-1)^k \frac{A_{k,2n}}{(2n+1) \binom{2n}{k}} \quad (1.16)$$

$$\mathcal{B}_{2n+1} = \sum_{k=0}^{2n+1} (-1)^k \frac{A_{k,2n+1}}{(2n+2) \binom{2n+1}{k}} = 0, \quad n \geq 1 \quad (1.17)$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \mathcal{B}_k = \mathcal{B}_{2n} \quad (1.18)$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \mathcal{B}_k = 0, \quad n \geq 1 \quad (1.19)$$

$$\sum_{k=0}^{n-1} \binom{2n}{2k} \mathcal{B}_{2k} = n, \quad n \geq 1 \quad (1.20)$$

$$\sum_{k=0}^n \binom{2n+1}{2k} \mathcal{B}_{2k} = n + \frac{1}{2}, \quad n \geq 1 \quad (1.21)$$

$$\sum_{k=1}^n \binom{2n}{2k-1} \mathcal{B}_{2k} = \frac{1}{2} - \mathcal{B}_{2n}, \quad n \geq 1 \quad (1.22)$$

$$\sum_{k=1}^n \binom{2n+1}{2k-1} \mathcal{B}_{2k} = \frac{1}{2}, \quad n \geq 1 \quad (1.23)$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \mathcal{B}_k = \mathcal{B}_{2n} + 2n, \quad n \geq 1 \quad (1.24)$$

2 Bernoulli Polynomials $\mathcal{B}_n(x)$

Remark 2.1 Throughout this chapter, we assume n , r , and p are nonnegative integers. We assume x, y, a, b , and t are real or complex numbers. Furthermore, for any real x , we let $[x]$ denote the floor of x .

2.1 Definition of $\mathcal{B}_n(x)$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{B}_k(x) = \frac{te^{xt}}{e^t - 1}, \quad |t| < 2\pi \quad (2.1)$$

2.1.1 Relationship to \mathcal{B}_n

$$\mathcal{B}_n = \mathcal{B}_n(0) \quad (2.2)$$

2.2 Alternative Definitions of $\mathcal{B}_n(x)$

$$\sum_{k=0}^{p-1} k^n = \frac{\mathcal{B}_{n+1}(p) - \mathcal{B}_{n+1}}{n+1}, \quad n, p \geq 1 \quad (2.3)$$

$$\mathcal{B}_{n+1}(x) = (n+1) \sum_{k=0}^n \binom{x}{k+1} B_{k,k}^n + \mathcal{B}_{n+1}, \quad \text{where } B_{k,k}^n = \sum_{j=0}^k (-1)^k \binom{k}{j} (k-j)^n \quad (2.4)$$

$$\mathcal{B}_{2n+1}(x) = (2n+1) \sum_{k=0}^{2n} \binom{x}{k+1} B_{k,k}^{2n}, \quad n \geq 1 \quad (2.5)$$

2.3 Explicit Formulas for $\mathcal{B}_n(x)$

$$\mathcal{B}_n(x) = \sum_{j=0}^n \frac{1}{j+1} \sum_{k=0}^j (-1)^k \binom{j}{k} (x+k)^n \quad (2.6)$$

$$\mathcal{B}_n(x) = \sum_{k=0}^n \binom{n}{k} x^{n-k} \mathcal{B}_k \quad (2.7)$$

2.3.1 Application of Equation (2.7)

$$\mathcal{B}_{n-j} = \frac{n}{\binom{n}{j}} \sum_{k=j-1}^{n-1} C_j^{k+1} B_{k,k}^{n-1}, \text{ where } 1 \leq j \leq n, \text{ and } \binom{x}{n} = \sum_{k=0}^n C_k^n x^k \quad (2.8)$$

2.4 Properties of $\mathcal{B}_n(x)$

2.4.1 Shift Property

$$\mathcal{B}_n(1-x) = (-1)^n \mathcal{B}_n(x) \quad (2.9)$$

Applications of Equation (2.9)

$$\mathcal{B}_n(1) = (-1)^n \mathcal{B}_n \quad (2.10)$$

$$\mathcal{B}_n = (-1)^n \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^2} B_{k,k}^{n+1} \quad (2.11)$$

2.4.2 Addition Property

$$\mathcal{B}_n(x+y) = \sum_{k=0}^n \binom{n}{k} x^{n-k} \mathcal{B}_k(y) \quad (2.12)$$

Application of Equation (2.12)

$$(-1)^n \mathcal{B}_n(y-1) = \sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{B}_k(y) \quad (2.13)$$

2.4.3 Appell Derivative Property

$$\frac{d}{dx}\mathcal{B}_n(x) = \begin{cases} n\mathcal{B}_{n-1}(x), & n \geq 1 \\ 0, & n = 0 \end{cases} \quad (2.14)$$

Applications of Equation (2.14)

Remark 2.2 Recall that $D_x^r f(x)$ is the r^{th} derivative of $f(x)$ with respect to x .

$$D_x^r \mathcal{B}_n(ax) = r! \binom{n}{r} a^r \mathcal{B}_{n-r}(ax) \quad (2.15)$$

$$D_x \mathcal{B}_n(ax + b) = a \cdot n \cdot \mathcal{B}_{n-1}(ax + b) \quad (2.16)$$

2.4.4 Integration of $\mathcal{B}_n(x)$

$$\int \mathcal{B}_n(x) dx = \frac{1}{n+1} \mathcal{B}_{n+1}(x) + C \quad (2.17)$$

Applications of Equation (2.17)

$$\int \mathcal{B}_n(ax) dx = \frac{1}{a(n+1)} \mathcal{B}_{n+1}(ax) + C, \quad a \neq 0 \quad (2.18)$$

$$\int_0^1 \mathcal{B}_n(x) dx = \begin{cases} 0, & n \geq 1 \\ 1, & n = 0 \end{cases} \quad (2.19)$$

2.4.5 Other Integrals Involving $\mathcal{B}_n(x)$

$$\int_0^1 x^j \mathcal{B}_n(x) dx = \frac{(-1)^n}{\binom{n+j}{n}} \sum_{k=0}^n \binom{n+j}{n-k} \frac{\mathcal{B}_{n-k}}{k+j+1} \quad (2.20)$$

$$\int_0^1 x^j \mathcal{B}_n(x) dx = (-1)^n \sum_{k=0}^n \binom{n}{k} \frac{\mathcal{B}_{n-k}}{\binom{k+j}{k} (k+j+1)} \quad (2.21)$$

$$\int_0^1 x^j \mathcal{B}_n(x) dx = \sum_{k=0}^n \binom{n}{k} \frac{\mathcal{B}_{n-k}}{k+j+1} \quad (2.22)$$

$$\int_0^1 x \mathcal{B}_n(x) dx = (-1)^{n+1} \frac{\mathcal{B}_{n+1}}{n+1} \quad (2.23)$$

$$\int_0^1 \mathcal{B}_n(x) \mathcal{B}_r(x) dx = (-1)^{r-1} \frac{\mathcal{B}_{r+n}}{\binom{r+n}{n}} = (-1)^{n-1} \frac{\mathcal{B}_{r+n}}{\binom{r+n}{n}}, \quad r \geq 1, n \geq 1 \quad (2.24)$$

2.4.6 Convolution Properties

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{B}_k(x) \mathcal{B}_{n-k}(x) = (-1)^{n-1} (n-1) \mathcal{B}_n \quad (2.25)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{B}_k(x) \mathcal{B}_{n-k}(x) = (1-n) \mathcal{B}_n \quad (2.26)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{B}_k \mathcal{B}_{n-k} = (1-n) \mathcal{B}_n \quad (2.27)$$

$$\sum_{k=0}^n \binom{2n}{2k} \mathcal{B}_{2k} \mathcal{B}_{2n-2k} = (1-2n) \mathcal{B}_{2n}, \quad n \geq 2 \quad (2.28)$$

$$\sum_{k=1}^{n-1} \binom{2n}{2k} \mathcal{B}_{2k} \mathcal{B}_{2n-2k} = -(2n+1) \mathcal{B}_{2n}, \quad n \geq 2 \quad (2.29)$$

2.4.7 A Binomial Expansion

Remark 2.3 *In the following identity, we let $\mathcal{B}(x)$ denote the Bernoulli Polynomial, and assume*

$$\binom{\mathcal{B}(x)}{r} \equiv \sum_{k=0}^r C_k^r \mathcal{B}_k(x). \quad (2.30)$$

Then,

$$\binom{\mathcal{B}(x) + n}{n} = \sum_{k=0}^n (-1)^k \binom{n+x}{n-k} \frac{1}{k+1}. \quad (2.31)$$

Also,

$$\binom{\mathcal{B} + n}{n} = \frac{1}{n+1}. \quad (2.32)$$

2.5 Formulas Involving n^{th} Differences

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{B}_j(k) = (-1)^n \sum_{k=0}^j \binom{j}{k} \mathcal{B}_{j-k} B_{n,n}^k \quad (2.33)$$

$$\Delta_{x,1} \mathcal{B}_n(x) \equiv \mathcal{B}_n(x+1) - \mathcal{B}_n(x) = nx^{n-1}, \quad n \geq 0, \quad x \neq 0 \quad (2.34)$$

2.5.1 Applications of Equation (2.34)

$$x^n = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} \mathcal{B}_k(x) \quad (2.35)$$

$$\sum_{k=0}^r (-1)^k k^{n-1} = \frac{1}{n} \left((-1)^r \mathcal{B}_n(r+1) - \mathcal{B}_n + 2 \sum_{k=0}^r (-1)^{k-1} \mathcal{B}_n(k) \right), \quad n \geq 1 \quad (2.36)$$

2.6 Polynomial Expansions Involving $\mathcal{B}_n(x)$

Remark 2.4 Throughout this section, we assume $f(x)$ is a polynomial of degree n , namely,

$$f(x) = \sum_{k=0}^n a_k x^k. \quad (2.37)$$

2.6.1 Basic Expansion Formulas

Remark 2.5 The following expansion is equivalent to the formula given by Charles Jordan on Page 248 of Calculus of Finite Differences, Chelsea Publishing, New York, 1947.

$$f(x) = \sum_{j=0}^n \mathcal{B}_j(x) \sum_{k=j}^n \binom{k+1}{j} \frac{D^k f(0)}{(k+1)!} \quad (2.38)$$

$$f(x) = \sum_{k=0}^n \mathcal{B}_k(x) C_k, \quad \text{where } C_0 = \int_0^1 f(x) dx, \quad C_k = \frac{1}{k!} \Delta_{x,1} D^{k-1} f(x)|_{x=0} \quad (2.39)$$

2.6.2 Raabe's Theorem

Remark 2.6 *The identities in this section are found on Page 252 of Charles Jordan's Calculus of Finite Differences.*

$$\mathcal{B}_n(x) = r^{n-1} \sum_{k=0}^{r-1} \mathcal{B}_n\left(\frac{x+k}{r}\right), \quad r \geq 1 \quad (2.40)$$

$$\mathcal{B}_n(rx) = r^{n-1} \sum_{k=0}^{r-1} \mathcal{B}_n\left(x + \frac{k}{r}\right), \quad r \geq 1 \quad (2.41)$$

Applications of Equation (2.40)

$$\mathcal{B}_n = r^{n-1} \sum_{k=0}^{r-1} \mathcal{B}_n\left(\frac{k}{r}\right), \quad r \geq 1 \quad (2.42)$$

$$\mathcal{B}_n\left(\frac{1}{2}\right) = -\left(1 - \frac{1}{2^{n-1}}\right) \mathcal{B}_n \quad (2.43)$$

$$\mathcal{B}_n\left(\frac{1}{3}\right) + \mathcal{B}_n\left(\frac{2}{3}\right) = -\left(1 - \frac{1}{3^{n-1}}\right) \mathcal{B}_n \quad (2.44)$$

2.6.3 Generalizations of Equation (2.39)

Remark 2.7 *Two excellent reference for the formulas found in this subsection are Konrad Knopp's Theorie und Anwendung der unendlichen Reihen, fourth edition, Berlin, 1947, and N. E. Nörlund's Vorlesungen über Differenzenrechnung, Berlin, 1924 (Chelsea Reprint, New York 1954).*

Let w be a nonzero real or complex number. Then,

$$f(x+wz) = \frac{1}{w} \int_x^{x+w} f(t) dt + \sum_{k=1}^n \frac{w^k}{k!} \mathcal{B}_k(z) \cdot \Delta_{x,w} D_x^{k-1} f(x), \quad (2.45)$$

$$\text{where } \Delta_{x,w} f(x) \equiv \frac{f(x+w) - f(x)}{w}.$$

Applications of Equation (2.45)

$$f(x+z) = \int_x^{x+1} f(t) dt + \sum_{k=1}^n \frac{\mathcal{B}_k(z)}{k!} \cdot \Delta_{x,1} D_x^{k-1} f(x) \quad (2.46)$$

$$D_x f(x+z) = \sum_{k=0}^{n-1} \frac{\mathcal{B}_k(z)}{k!} \cdot \Delta_{x,1} D_x^k f(x) \quad (2.47)$$

2.6.4 Euler-Maclaurin Formula

Remark 2.8 For this subsection, we define $\tilde{\mathcal{B}}_n(x)$ to be the periodic real valued function which agrees with $\mathcal{B}_n(x)$ on the interval $0 \leq x < 1$. Furthermore, we do not require $f(x)$ be a polynomial, only that $f(x)$ be sufficiently smooth.

Euler-Maclaurin Formula

$$\begin{aligned} \sum_{k=0}^n f(k) &= \int_0^n f(x) dx + \frac{f(0) + f(n)}{2} \\ &+ \sum_{k=1}^j \frac{\mathcal{B}_{2k}}{(2k)!} (D_x^{2k-1} f(x)|_{x=n} - D_x^{2k-1} f(x)|_{x=0}) + R_j, \end{aligned} \quad (2.48)$$

where,

$$R_j = \frac{1}{(2j+1)!} \int_0^n \tilde{\mathcal{B}}_{2j+1}(x) \cdot D_x^{2j+1} f(x) dx \quad (2.49)$$

Applications of Equation (2.48)

$$\begin{aligned} \log \Gamma(z) &= \left(z - \frac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + \sum_{k=1}^j \frac{\mathcal{B}_{2k}}{2k(2k-1)z^{2k-1}} \\ &- \frac{1}{2j+1} \int_0^\infty \frac{\tilde{\mathcal{B}}_{2j+1}(x)}{(z+x)^{2j+1}} dx \end{aligned} \quad (2.50)$$

$$\begin{aligned} \log n! &= \left(n + \frac{1}{2}\right) \log n - n + \log \sqrt{2\pi} + \sum_{k=1}^j \frac{\mathcal{B}_{2k}}{2k(2k-1)n^{2k-1}} \\ &- \frac{1}{2j+1} \int_0^\infty \frac{\tilde{\mathcal{B}}_{2j+1}(x)}{x^{2j+1}} dx \end{aligned} \quad (2.51)$$

$$\sum_{k=1}^n \frac{1}{k} = \log n + \frac{1}{2} + \frac{1}{2n} + \sum_{k=1}^j \frac{\mathcal{B}_{2k}}{2k} \left(1 - \frac{1}{n^{2k}}\right) - \int_1^n \frac{\tilde{\mathcal{B}}_{2j+1}(x)}{x^{2j+2}} dx, \quad n \geq 1 \quad (2.52)$$

3 Eulerian Polynomials $E_n(x)$ and Eulerian Numbers E_n

Remark 3.1 Throughout this chapter, we assume $n, r, m,$ and p are nonnegative integers. We assume $x, y,$ and t are real or complex numbers. Furthermore, for any real x , we let $[x]$ denote the floor of x .

3.1 Definition of $E_n(x)$

$$\sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} = \frac{2e^{xt}}{e^t + 1} \quad (3.1)$$

3.1.1 Relationships Between $E_n(x)$ and $\mathcal{B}_n(x)$

$$E_{n-1}(x) = \frac{2^n}{n} \left(\mathcal{B}_n \left(\frac{x+1}{2} \right) - \mathcal{B}_n \left(\frac{x}{2} \right) \right), \quad n \geq 1 \quad (3.2)$$

$$E_{n-1}(x) = \frac{2}{n} \left(\mathcal{B}_n(x) - 2^n \mathcal{B}_n \left(\frac{x}{2} \right) \right), \quad n \geq 1 \quad (3.3)$$

3.2 Properties of $E_n(x)$

3.2.1 Appell Derivative Property

Remark 3.2 Throughout this chapter, we let $D_x^r f(x)$ denote the r^{th} derivative of $f(x)$.

$$D_x E_n(x) = \begin{cases} n E_{n-1}(x), & n \geq 1 \\ 0, & n = 0 \end{cases} \quad (3.4)$$

Applications of Equation (3.4)

$$D_x^r E_n(x) = r! \binom{n}{r} E_{n-r}(x) \quad (3.5)$$

$$\int E_n(x) dx = \frac{E_{n+1}(x)}{n+1} + C \quad (3.6)$$

$$\int_a^b E_n(x) dx = \frac{E_{n+1}(b) - E_{n+1}(a)}{n+1} \quad (3.7)$$

$$\int_0^1 E_n(x) dx = \frac{(-1)^{n+1} - 1}{n+1} E_{n+1}(0) \quad (3.8)$$

3.2.2 Addition Property

$$E_n(x + y) = \sum_{k=0}^n \binom{n}{k} x^k E_{n-k}(y) \quad (3.9)$$

Applications of Equation (3.9)

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} x^k E_{n-k}(0) \quad (3.10)$$

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} x^{n-k} \frac{2(1 - 2^{k+1})}{k + 1} \mathcal{B}_{k+1} \quad (3.11)$$

3.2.3 Shift Property

$$E_n(x) = (-1)^n E_n(1 - x) \quad (3.12)$$

3.2.4 Difference Equation

$$E_n(x) + E_n(x + 1) = 2x^n \quad (3.13)$$

Application of Equation (3.13)

$$\sum_{k=0}^n (-1)^k k^p = \frac{(-1)^n E_p(n + 1) + E_p(0)}{2} \quad (3.14)$$

3.2.5 An Integral Property

$$\int_0^1 E_m(x) E_n(x) dx = -\frac{(-1)^m + (-1)^n}{m + n + 1} \cdot \frac{E_{m+n+1}(0)}{\binom{m+n}{n}} \quad (3.15)$$

Application of Equation (3.15)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} E_k(x) E_{n-k}(x) = ((-1)^{n+1} - 1) E_{n+1}(0) \quad (3.16)$$

3.3 Explicit Formula for $E_n(x)$

$$E_n(x) = \sum_{j=0}^n \frac{1}{2^j} \sum_{k=0}^j (-1)^k \binom{j}{k} (x + k)^n \quad (3.17)$$

3.3.1 Calculations of $E_n(0)$ and \mathcal{B}_n

$$E_n(0) = \frac{1}{2^n} \sum_{k=0}^n \binom{n+1}{n-k} \sum_{j=0}^k (-1)^j j^n, \quad n \geq 1 \quad (3.18)$$

$$E_n(0) = \frac{1}{2^n} \sum_{k=0}^n (-1)^k A_{k,n}, \quad \text{where } A_{k,n} = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k-j)^n \quad (3.19)$$

$$E_n(0) = \frac{1}{2^n} \sum_{k=1}^n (-1)^k A_{k,n}, \quad n \geq 1 \quad (3.20)$$

$$E_n(0) = \sum_{k=0}^n \frac{(-1)^k}{2^k} B_{k,k}^n \quad \text{where } B_{k,k}^n = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n \quad (3.21)$$

$$\mathcal{B}_{n+1} = \frac{n+1}{(2^{n+1}-1)} \sum_{k=0}^n \frac{(-1)^{k+1}}{2^{k+1}} B_{k,k}^n \quad (3.22)$$

$$\mathcal{B}_{n+1} = \frac{n+1}{2^{n+1}(2^{n+1}-1)} \sum_{k=0}^n (-1)^{k+1} A_{k,n} \quad (3.23)$$

3.4 Definition of E_n

$$\frac{1}{\cos z} = \sum_{k=0}^{\infty} E_k \frac{i^k z^k}{k!}, \quad i \equiv \sqrt{-1}, \quad |z| < \frac{\pi}{2} \quad (3.24)$$

3.4.1 Relationship Between E_n and $E_n(x)$

$$E_n = 2^n E_n \left(\frac{1}{2} \right) \quad (3.25)$$

3.5 Properties of E_n

3.5.1 Parity Properties

$$E_{2n+1} = 0 \quad (3.26)$$

$$E_0 = 1 \quad (3.27)$$

$$\sum_{k=0}^n \binom{2n}{2k} E_{2k} = \begin{cases} 0, & n \geq 1 \\ 1, & n = 0 \end{cases} \quad (3.28)$$

$$E_{2n} = 1 + \sum_{k=1}^n \binom{2n}{2k-1} \frac{2^{2k}(1-2^{2k})}{2k} \mathcal{B}_{2k}, \quad n \geq 1 \quad (3.29)$$

3.5.2 Explicit Formula for $E_n(x)$

$$E_n(z) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(z - \frac{1}{2}\right)^{n-k} \quad (3.30)$$

3.5.3 Eulerian Numbers in Various Trigonometric Expansions

$$\pi \sec \pi z = \pi \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k}}{(2k)!} \pi^{2k} z^{2k}, \quad |z| < \frac{1}{2} \quad (3.31)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n \frac{\pi^{2n+1} E_{2n}}{2^{2n+2} (2n)!} \quad (3.32)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{(-1)^n \pi^{2n+1}}{(2n)!} \sum_{j=0}^{2n} \frac{1}{2^{2n+2+j}} \sum_{k=0}^j (-1)^j \binom{j}{k} (2k+1)^{2n} \quad (3.33)$$

4 Generalized Bernoulli Polynomials $\mathcal{B}_k^{(n)}(x)$

Remark 4.1 Throughout this chapter, we assume $n, r, m,$ and p are nonnegative integers. We assume $x, y, a, b,$ and t are real or complex numbers. Furthermore, for any real $x,$ we let $[x]$ denote the floor of $x.$

4.1 Definition of $\mathcal{B}_k^{(n)}(x)$

$$\sum_{k=0}^{\infty} \mathcal{B}_k^{(n)}(x) \frac{z^k}{k!} = \frac{z^n e^{xz}}{(e^z - 1)^n}, \quad |z| < 2\pi \quad (4.1)$$

$$\mathcal{B}_k(x) = \mathcal{B}_k^{(1)}(x), \quad \mathcal{B}_k^{(1)}(0) = \mathcal{B}_k(0) = \mathcal{B}_k \quad (4.2)$$

4.1.1 The Reciprocal Generating Function of Equation (4.1)

$$\frac{(e^x - 1)^n}{e^{tx} x^n} = n! \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \frac{B_{n,n}^{j+n}}{\binom{j+n}{j}} t^{k-j}, \quad \text{where } B_{k,k}^n = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n \quad (4.3)$$

4.1.2 Generating Function for $\mathcal{B}_k^{(k-n+1)}(t)$: Formula 2247 Volume 5

$$\frac{(1+x)^{t-1} x^n}{(\ln(1+x))^n} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \mathcal{B}_k^{(k-n+1)}(t) \quad (4.4)$$

4.2 Derivative Definition of $\mathcal{B}_k^{(n)}(x)$

Remark 4.2 Throughout this section, and the remainder of this chapter, we let $D_x^r f(x)$ denote the r^{th} derivative of $f(x)$ with respect to $x.$

$$\mathcal{B}_k^{(n+1)}(z+1) = k! D_z^{n-k} \binom{z}{n} \quad (4.5)$$

4.2.1 Applications of Equation (4.5)

$$C_{n-k}^n = \frac{1}{k!(n-k)!} \mathcal{B}_k^{(n+1)}(1), \quad \text{where } \binom{x}{n} = \sum_{k=0}^n C_k^n x^k \quad (4.6)$$

$$\mathcal{B}_n^{(n+1)}(z+1) = n! \binom{z}{n} \quad (4.7)$$

$$\mathcal{B}_n^{(n+1)}(z) = \prod_{k=1}^n (z-k), \quad n \geq 1 \quad (4.8)$$

4.3 Generalized Bernoulli Numbers $\mathcal{B}_k^{(n)} \equiv \mathcal{B}_k^{(n)}(0)$

4.3.1 Relationships with C_{n-k}^n

$$\mathcal{B}_k^{(n)} = \frac{n!}{\binom{n-1}{k}} C_{n-k}^n \quad (4.9)$$

$$n!k!C_n^{n+k} = \frac{n}{n+k} \mathcal{B}_k^{(n+k)} \quad (4.10)$$

4.3.2 Exponential Generating Function

$$\left(\frac{\ln(1+x)}{x} \right)^n = n \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{\mathcal{B}_k^{(n+k)}}{(n+k)}, \quad n \geq 1, \quad |x| < 1 \quad (4.11)$$

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \mathcal{B}_k^{(k)}, \quad |x| < 1 \quad (4.12)$$

4.3.3 Relationships with $B_{n,n}^{k+n}$

$$\mathcal{B}_k^{(n)} = \sum_{j=0}^k (-1)^j \binom{k+1}{j+1} \frac{B_{jn,jn}^{k+jn}}{(jn)! \binom{k+jn}{k}} \quad (4.13)$$

$$\mathcal{B}_k^{(-n)} = \frac{k!}{(k+n)!} B_{n,n}^{k+n} \quad (4.14)$$

$$(-1)^n \frac{(n+j)!}{j!} \mathcal{B}_j^{(-n)} = \sum_{k=0}^n (-1)^k \binom{n}{k} k^{n+j} \quad (4.15)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k^{n+j} = (-1)^n \frac{(n+j)!}{j!} \mathcal{B}_j^{(-n)} = (-1)^n B_{n,n}^{n+j} \quad (4.16)$$

4.3.4 A Generalization of Generalized Bernoulli Numbers $\mathcal{B}_k^{(z)}$

$$\mathcal{B}_k^{(z)} = \sum_{j=0}^k (-1)^j \binom{z+k}{k-j} \binom{z+j-1}{j} \frac{B_{j,j}^{j+k}}{j! \binom{j+k}{j}} \quad (4.17)$$

Applications of Equation (4.17)

$$\mathcal{B}_k^{(n)} = \sum_{j=0}^k (-1)^j \binom{n+k}{k-j} \binom{n+j-1}{j} \frac{B_{j,j}^{j+k}}{j! \binom{j+k}{j}} \quad (4.18)$$

$$\mathcal{B}_k^{(-n)} = \sum_{j=0}^k (-1)^j \binom{k-n}{k-j} \binom{-n+j-1}{j} \frac{B_{j,j}^{j+k}}{j! \binom{j+k}{j}} \quad (4.19)$$

$$\mathcal{B}_k^{(-z)} = \sum_{j=0}^k \binom{k-z}{k-j} \binom{z}{j} \frac{B_{j,j}^{j+k}}{j! \binom{j+k}{j}} \quad (4.20)$$

$$\left(\frac{x}{e^x - 1} \right)^z = \sum_{k=0}^{\infty} \frac{x^k}{k!} \mathcal{B}_k^{(z)}, \quad |x| < 2\pi \quad (4.21)$$

$$\frac{t^z e^{xt}}{(e^t - 1)^z} = \sum_{k=0}^{\infty} \mathcal{B}_k^{(z)}(x) \frac{t^k}{k!}, \quad |t| < 2\pi \quad (4.22)$$

Restatement of a Theorem of Schläfli's

$$\mathcal{B}_k^{(z)} = \sum_{j=0}^k \binom{-z}{j} \binom{k+z}{k-j} \mathcal{B}_k^{(-j)} \quad (4.23)$$

4.4 Properties of $\mathcal{B}_k^{(n)}(x)$

4.4.1 Addition Property

$$\mathcal{B}_k^{(n)}(x+y) = \sum_{j=0}^k \binom{k}{j} \mathcal{B}_j^{(n)}(x) y^{k-j} \quad (4.24)$$

Application of Equation (4.24)

$$\mathcal{B}_j^{(n+1)}(x+1) = \sum_{k=0}^j \binom{j}{k} x^{j-k} \sum_{J=0}^k \binom{k}{J} \sum_{r=0}^J (-1)^r \binom{J+1}{r+1} \frac{B_{rn+r, rn+r}^{J+rn+r}}{(rn+r)! \binom{J+rn+r}{J}} \quad (4.25)$$

4.4.2 Appell Derivative Property

$$D_x \mathcal{B}_k^{(n)}(x) = k \mathcal{B}_{k-1}^{(n)}(x), \quad k \geq 1 \quad (4.26)$$

Application of Equation (4.26)

$$D_x^j \mathcal{B}_k^{(n)}(x) = \frac{k!}{(k-j)!} \mathcal{B}_{k-j}^{(n)}(x) \quad (4.27)$$

4.4.3 n^{th} Difference Formula

$$\Delta_{x,1} \mathcal{B}_k^{(n)}(x) = \begin{cases} k \mathcal{B}_{k-1}^{(n-1)}(x), & k \geq 1 \\ 0, & k = 0 \end{cases}, \quad (4.28)$$

$$\text{where } \Delta_{x,w} f(x) \equiv \frac{f(x+w) - f(x)}{w}$$

Applications of Equation (4.28)

$$\Delta_{x,1}^r \mathcal{B}_k^{(n)}(x) = r! \binom{k}{r} \mathcal{B}_{k-r}^{(n-r)}(x) \quad (4.29)$$

$$\Delta_{x,1} D_x^k \binom{x}{n} = \frac{1}{(n-k-1)!} \mathcal{B}_{n-k-1}^{(n)}(x+1) \quad (4.30)$$

Newton Expansion

$$\mathcal{B}_k^{(z)}(x+y) = \sum_{j=0}^k \binom{x}{j} \binom{k}{j} j! \mathcal{B}_{k-j}^{(z-j)}(y) \quad (4.31)$$

4.4.4 Difference Relation

$$k\mathcal{B}_k^{(n)}(t) = n\mathcal{B}_k^{(n)}(t) + tk\mathcal{B}_{k-1}^{(n)}(t) - n\mathcal{B}_k^{(n+1)}(t+1), \quad k \geq 1 \quad (4.32)$$

Applications of Equation (4.32)

$$\mathcal{B}_k^{(n+1)}(t) = \left(1 - \frac{k}{n}\right) \mathcal{B}_k^{(n)}(t) + (t-n)\frac{k}{n}\mathcal{B}_{k-1}^{(n)}(t), \quad k \geq 1, \quad n \geq 1 \quad (4.33)$$

$$\mathcal{B}_n^{(n+1)}(x) = (x-n)\mathcal{B}_{n-1}^{(n)}(x), \quad n \geq 1 \quad (4.34)$$

$$\mathcal{B}_n^{(n+1)}(x) = \prod_{k=0}^j (x-n-k) \cdot \mathcal{B}_{n-j-1}^{(n-j)}(x) \quad (4.35)$$

$$\mathcal{B}_k^{(n+1)}(1) = \left(1 - \frac{k}{n}\right) \mathcal{B}_k^{(n)}, \quad n \geq 1 \quad (4.36)$$

4.4.5 An Integral Involving $\mathcal{B}_k^{(n)}(t)$

$$\int_x^{x+1} \binom{z}{n} dz = \frac{1}{n!} \mathcal{B}_n^{(n)}(x+1) \quad (4.37)$$

Applications of Equation (4.37)

$$\sum_{k=0}^n \binom{x}{n-k} \frac{1}{k!} \mathcal{B}_k^{(k)}(z+1) = \frac{1}{n!} \mathcal{B}_n^{(n)}(z+x+1) \quad (4.38)$$

$$(-1)^n \mathcal{B}_n^{(n)} = \sum_{k=0}^n \binom{n}{k} \frac{1}{k!} \mathcal{B}_k^{(k)} \quad (4.39)$$

4.4.6 Convolution Property

$$\mathcal{B}_k^{(z+w)}(x+y) = \sum_{j=0}^k \binom{k}{j} \mathcal{B}_j^{(z)}(x) \mathcal{B}_{k-j}^{(w)}(y) \quad (4.40)$$

Applications of Equation (4.40)

$$(x+y)^k = \sum_{j=0}^k \binom{k}{j} \mathcal{B}_j^{(z)}(x) \mathcal{B}_{k-j}^{(-z)}(y) \quad (4.41)$$

$$x^k = \sum_{j=0}^k \binom{k}{j} \mathcal{B}_{k-j}^{(n)}(x) \frac{B_{n,n}^{j+n}}{n! \binom{j+n}{j}} \quad (4.42)$$

$$y^k = \sum_{j=0}^k \binom{k}{j} \mathcal{B}_{k-j}^{(-n)} \frac{n!}{\binom{n-1}{j}} C_{n-j}^n \quad (4.43)$$

4.5 Polynomial Expansions Using Generalized Bernoulli and Euler Polynomials

Remark 4.3 *The identities of this section are found in Chapters 6 and 9 of N. E. Nörlund's Vorlesungen über Differenzenrechnung, Berlin, 1924 (reprinted by Chelsea Publ. Co., New York, 1954).*

Remark 4.4 *Throughout this section, we assume, unless otherwise stated, that $f(x)$ is a polynomial of degree m , namely, $f(x) = \sum_{k=0}^m a_k x^k$.*

4.5.1 Definition of General Euler Polynomials $E_k^{(n)}(t)$

$$\sum_{k=0}^{\infty} E_k^{(n)}(t) \frac{x^k}{k!} = \frac{2^n e^{tx}}{(e^x + 1)^n} \quad (4.44)$$

4.5.2 Generalized Bernoulli and Euler Expansions

Remark 4.5 *Throughout this section, we will use the averaging operator $\nabla_{x,w} f(x) \equiv \frac{f(x+w)+f(x)}{2}$.*

$$D_t^n f(t)|_{t=x+z} \equiv f^{(n)}(x+z) = \sum_{k=0}^m \mathcal{B}_k^{(n)}(z) \frac{1}{k!} \Delta_{x,1}^n D_x^k f(x) \quad (4.45)$$

$$f^{(n)}(x+z) = \sum_{k=0}^m E_k^{(n)}(z) \frac{1}{k!} \nabla_{x,1}^n D_x^k f(x), \quad (4.46)$$

where $f(x)$ is a polynomial of degree $n+m$

4.5.3 Newton Series Involving $\mathcal{B}_k^{(k+2)}$

Recall that the Newton Series for $f(x + y)$ is

$$f(x + y) = \sum_{k=0}^m \binom{\frac{x}{z}}{k} z^k \Delta_{y,z}^k f(y). \quad (4.47)$$

Alternate Form of the Newton Series

$$f(x + y) = \sum_{k=0}^m \binom{\frac{y-z}{z}}{k} z^k \Delta_{x,z}^k f(x + z) \quad (4.48)$$

Derivative Applications of Equation (4.48)

$$D_y f(x + y) = \sum_{k=1}^m \binom{\frac{y-z}{z}}{k} z^k \sum_{j=1}^k \frac{1}{y - jz} \Delta_{x,z}^k f(x + z), \quad m \geq 1 \quad (4.49)$$

$$f'(x) = \sum_{k=0}^{m-1} (-1)^k z^k \sum_{j=0}^k \frac{1}{j+1} \Delta_{x,z}^{k+1} f(x + z), \quad m \geq 1 \quad (4.50)$$

$$f'(x) = \sum_{k=0}^{m-1} \frac{z^k}{k!} \mathcal{B}_k^{(k+2)} \Delta_{x,z}^{k+1} f(x + z), \quad m \geq 1 \quad (4.51)$$

$$f'(x) = \sum_{k=0}^{m-1} (-1)^k \frac{z^k}{k+1} \Delta_{x,z}^{k+1} f(x), \quad m \geq 1 \quad (4.52)$$

$$D_x^j f(x) = \sum_{k=0}^{m-j} \frac{z^k}{k!} \mathcal{B}_k^{(k+j+1)} \Delta_{x,z}^{k+j} f(x + z) \quad (4.53)$$

$$D_x^j f(x) = j \sum_{k=0}^{m-j} \frac{z^k}{k+j} \frac{1}{k!} \mathcal{B}_k^{(k+j)} \Delta_{x,z}^{k+j} f(x), \quad j \geq 1 \quad (4.54)$$

Remark 4.6 For the remaining identities of this section, we do not require that $f(x)$ be a polynomial.

$$\frac{1}{x^r} = \sum_{k=0}^{\infty} (-1)^k \frac{\binom{r-1+k}{k}}{\binom{x+r+k}{r+k}} \frac{1}{(r+k)!} \mathcal{B}_k^{(r+k)}, \quad \operatorname{Re}(x) > 0, \quad r \geq 1 \quad (4.55)$$

$$\frac{1}{(x-y)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} \frac{\mathcal{B}_{k-n}^{(k+1)}(y+k)}{x(x+1)(x+2)\dots(x+k)}, \quad \operatorname{Re}(x) > \operatorname{Re}(y) \quad (4.56)$$

$$\frac{1}{z} \int_z^{x+z} f(t) dt = \sum_{k=0}^{\infty} z^k \frac{1}{k!} \mathcal{B}_k^{(k)} \Delta_{x,z}^k f(x+z) \quad (4.57)$$

$$\frac{1}{z} \int_z^{x+z} f(t) dt = f(x) + \frac{z}{2} \Delta_{x,z} f(x) - \sum_{k=2}^{\infty} \frac{z^k}{(k-1)k!} \mathcal{B}_k^{(k-1)} \Delta_{x,z}^k f(x) \quad (4.58)$$

$$\log \left(1 + \frac{1}{x} \right) = \sum_{k=0}^{\infty} (-1)^k \frac{\mathcal{B}_k^{(k)}}{k! \binom{x+k}{k}}, \quad \operatorname{Re}(x) > 0 \quad (4.59)$$

$$\log \left(1 + \frac{1}{x} \right) = \frac{1}{x} - \frac{1}{2x(x+1)} - \sum_{k=2}^{\infty} (-1)^k \frac{\mathcal{B}_k^{(k-1)}}{(k-1)k! x \binom{x+k}{k}}, \quad \operatorname{Re}(x) > 0 \quad (4.60)$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{\mathcal{B}_k^{(k)}}{k \cdot k! \binom{x+k}{k}}, \quad \operatorname{Re}(x) > 0 \quad (4.61)$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} - \sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} \frac{\mathcal{B}_k^{(k-1)}}{k! \binom{x+k-1}{k}}, \quad \operatorname{Re}(x) > 0 \quad (4.62)$$

$$D_x^n \left(\frac{\Gamma'(x)}{\Gamma(x)} \right) = (-1)^{n-1} (n-1)! \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} \cdot \frac{\mathcal{B}_k^{(k+n+1)}}{(x+1)(x+2)\dots(x+k+n)}, \quad \operatorname{Re}(x) > 0 \quad (4.63)$$

$$D_x^n \left(\frac{\Gamma'(x)}{\Gamma(x)} \right) = (-1)^{n-1} n! \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} \cdot \frac{\mathcal{B}_k^{(k+n)}}{x(x+1)(x+2)\dots(x+k+n)(k+n)}, \quad \operatorname{Re}(x) > 0 \quad (4.64)$$

4.5.4 Functional Expansions Involving Generalized Bernoulli Polynomials

Remark 4.7 For the identities of this section, we do not require $f(x)$ to be a polynomial.

$$f(x+y) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{x}{x+bn} \mathcal{B}_n^{(n+1)} \left(\frac{x+bn}{z} + 1 \right) \Delta_{t,z}^n f(t)|_{t=y-bn} \quad (4.65)$$

$$f(x+y) - f(y) = x \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \mathcal{B}_n^{(n+1)} \left(\frac{x+bn+b}{z} \right) \Delta_{t,z}^{n+1} f(t)|_{t=y-bn-b} \quad (4.66)$$

$$f'(y) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \mathcal{B}_n^{(n+1)} \left(\frac{bn+b}{z} \right) \Delta_{t,z}^{n+1} f(t)|_{t=y-bn-b} \quad (4.67)$$

5 Catalan Numbers c_n

Remark 5.1 Throughout this chapter, we assume n is a nonnegative integer.

5.1 Definition of c_n

$$c_n = \frac{1}{n+1} \binom{2n}{n} \quad (5.1)$$

5.2 Shifted Catalan Numbers a_n

$$a_n = \frac{1}{2} \binom{2n}{n} \frac{1}{2n-1} = \frac{1}{2n-1} \binom{2n-1}{n}, \quad n \geq 1 \quad (5.2)$$

5.2.1 Properties of a_n

$$a_n = c_{n-1}, \quad n \geq 1 \quad (5.3)$$

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k}, \quad n \geq 2, \quad a_1 = a_2 = 1 \quad (5.4)$$

Remark 5.2 Recall that for x real, $[x]$ denotes the floor of x . We also define $a_0 \equiv -1$

$$\sum_{k=1}^n a_k a_{n-k} = 0, \quad n \geq 2 \quad (5.5)$$

$$\sum_{k=0}^n a_k a_{n-k} = -a_n, \quad n \geq 2 \quad (5.6)$$

$$\sum_{k=0}^n a_k a_{n-k} = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_k a_{n-k} + \frac{1 + (-1)^n}{2} a_{\lfloor \frac{n}{2} \rfloor}^2 \quad (5.7)$$

$$\sum_{k=1}^{n-1} a_k a_{2n-k} = \frac{1}{2} (a_{2n} - a_n^2), \quad n \geq 1 \quad (5.8)$$

5.2.2 Generating Function

$$\sum_{k=1}^{\infty} a_k z^k = \sum_{k=1}^{\infty} \binom{2k}{k} \frac{z^k}{2k-1} = 1 - (1-4z)^{\frac{1}{2}}, \quad |z| < \frac{1}{4} \quad (5.9)$$

6 Fibonacci Numbers F_n

Remark 6.1 Throughout this chapter, we assume n is a nonnegative integer.

6.1 Recursive Definition for f_n

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 3, \quad F_1 = F_2 = 1, \quad F_0 = 0 \quad (6.1)$$

6.2 Properties of F_n

6.2.1 Summation Formulas

$$\sum_{k=1}^n F_k = F_{n+2} - 1, \quad n \geq 1 \quad (6.2)$$

$$\sum_{k=1}^n F_k^2 = F_n \cdot F_{n+1}, \quad n \geq 1 \quad (6.3)$$

6.2.2 Power Recurrences

$$F_{n+1}^2 - F_{n-2}^2 = 4F_n F_{n-1}, \quad n \geq 3 \quad (6.4)$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2, \quad n \geq 1 \quad (6.5)$$

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3, \quad n \geq 2 \quad (6.6)$$

6.2.3 Determinant Property

$$F_n F_{n+3} - F_{n+1} F_{n+2} = (-1)^{n-1}, \quad n \geq 1 \quad (6.7)$$

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n, \quad n \geq 2 \quad (6.8)$$

6.3 Explicit Formulas for F_n

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right), \quad n \geq 1 \quad (6.9)$$

$$F_{2n+1} = \sum_{k=0}^n \binom{n+k}{2k} \quad (6.10)$$

6.4 Limit Calculations Involving F_n

Remark 6.2 Throughout this section, we assume r , a , and b are nonnegative integers.

$$\lim_{n \rightarrow \infty} \frac{F_n}{F_{n+r}} \equiv u_r = \frac{(-1)^r}{2} \left(F_{r-1} - \sqrt{5}F_r + F_{r+1} \right) \quad (6.11)$$

$$\lim_{n \rightarrow \infty} \frac{F_n}{F_{n+r}} \equiv u_r = \frac{(-1)^r}{2} \left(2F_{r-1} - (\sqrt{5} - 1)F_r \right) \quad (6.12)$$

6.4.1 Properties of u_r

$$u_a \cdot u_b = u_{a+b} \quad (6.13)$$

$$u_1^r = u_r \quad (6.14)$$

$$u_r + u_{r+1} = u_{r-1}, \quad r \geq 1 \quad (6.15)$$

$$\sum_{k=2}^r u_k = 1 - u_{r-1}, \quad r \geq 2 \quad (6.16)$$

$$\sum_{k=2}^r u_1^k = \frac{u_1^2}{1 - u_1} (1 - u_1^{r-1}), \quad r \geq 2 \quad (6.17)$$

6.5 Shifted Fibonacci Numbers f_n

6.5.1 Recursive Formula

$$f_{n+1} = f_n + f_{n-1}, \quad f_0 = f_1 = 1 \quad (6.18)$$

$$f_n = F_{n+1} \quad (6.19)$$

6.5.2 Explicit Formula

Remark 6.3 Let x be a real number. Recall that $[x]$ denotes the floor of x .

$$f_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \quad (6.20)$$