Journal of Combinatorial Optimization, 7, 237–245, 2003

A Tight Semidefinite Relaxation of the MAX CUT Problem

HONGWEI LIU liuhongweixu@163.net
SANYANG LIU
Department of Applied Mathematics, Xidian University, Xi'an 710071, People's Republic of China

FENGMIN XU
School of Science, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China

Received December 21, 2000; Revised December 21, 2000; Accepted March 25, 2002

Abstract. We obtain a tight semidefinite relaxation of the MAX CUT problem which improves several previous SDP relaxation in the literature. Not only is it a strict improvement over the SDP relaxation obtained by adding all the triangle inequalities to the well-known SDP relaxation, but also it satisfy Slater constraint qualification (strict feasibility).

Keywords: MAX CUT problem, semidefinite relaxation, cut polytope, metric polytope

1. Introduction

In the MAX CUT (MC) problem, we are given an undirected graph $G = (V, E)$ with vertex set V, edge set E and weighted adjacency matrix $A(G)$. We wish to partition the vertices into two sets V_- and V_+ so as to maximize the sum of the weights on edges with one end-point in V_- and one end-point in V_+. It is well known that MC problem is an NP-hard problem and has a variety of applications, e.g., in statistical physics and VLSI design (Barahona et al., 1988). Goemans and Williamson present an elegant application of semidefinite programming to this problem, which yields a 0.878-approximation algorithm (Goemans and Williamson, 1995). In Goemans and Williamson (1995), they use the SDP:

$$(\mathbf{SDP1}) \quad \begin{array}{rl} \mu_1^* = \max & \text{trace} QX \\ \text{s.t.} & X \in \varepsilon_n \end{array}$$

(the matrix Q and the elliptope ε_n are formally defined in Section 2). Two tighter SDP relaxations for MC, introduced in Anjos and Wolkowicz (1999, 2000), guarantee an improvement on the optimal value of SDP1. But they do not satisfy the Slater constraint qualification (see Anjos and Wolkowicz, 2000 for details).

In this paper we present a SDP relaxations (SDP2) for MC problem. By adding n rows to a feasible point Z of SDP2, we can obtain a $(t(n)+1)\times(t(n-1)+1)$ matrix P, whose $t(i)+1$ $(i = 1, \ldots, n)$ row is equal to the first row of Z. And furthermore, by adding n columns to the matrix P, we can obtain a $(t(n)+1)\times(t(n)+1)$ matrix Y, whose $(t(i)+1)$ column is equal to the first column of P, $(i = 1, \ldots, n)$. Where $t(n) = \frac{1}{2}n(n+1)$. It is easy to see that Y is a feasible point of SDP relaxations presented in Anjos and Wolkowicz (1999, 2000). For more details see Anjos and Wolkowicz (1999, 2000). Thus SDP2 is tightening of the SDP1 and SDP relaxations presented in Anjos and Wolkowicz (1999, 2000). Particularly, Not only do it preserves all the helpful properties of the latter, but also it satisfies the Slater constraint qualification (strict feasibility).

In the following section we introduce all notation and definition we will use. In Section 3 we present the main results. Finally some numerical results are presented in Section 4.

2. Notation and preliminaries

Let S^n denote the space of $n \times n$ symmetric matrices with the trace inner product $\langle A, B\rangle = \text{trace}AB$. We will work in the space $S^{t(n-1)+1}$ and for any matrix $Z \in S^{t(n-1)+1}$, $Z \succeq 0$ denotes that it is positive semidefinite. We now define the operators that we will use.

For $A \in S^n$, the vector $\text{diag}(A) \in R^n$ is the diagonal of A, while the adjoint operator $\text{Diag}(v) = \text{diag}^*(v)$ is the diagonal matrix with diagonal formed from the $v \in R^n$. Also, the symmetric vectorizing operator $s = \text{hsvec}(A) \in R^{t(n-1)}$, is formed (columnwise) from A while ignoring the lower triangular part of A. Its inverse is the symmetrizing matrix operator $A = \text{hsMat}(s)$ while every diagonal entry is one.

Let vertex set $V = \{1, \ldots, n\}$, $A(G)$ denote adjacency matrix of G. We assume that the graph in question is complete (if not, the non-edges can be given weight 0 to complete the graph). Let L denote the Laplacian matrix associated with the graph, hence $L = \text{Diag}(A(G)e) - A(G)$. Where e denotes the vector of ones. Let the $v \in \{-1, 1\}^n$ represent any cut, where $V_- = \{i : v_i = -1\}$ and $V_+ = \{i : v_i = +1\}$. Then the MC problem can be rephrased as follows.

$$\begin{aligned} \mu^* = \max \quad & \frac{1}{4} v^T L v \\ \text{s.t.} \quad & v \in \{-1, 1\}^n. \end{aligned}$$

Using $X = vv^T$, $v^T L v = \text{trace}LX$ and $Q = \frac{1}{4}L$, an equivalent formulation is

$$\begin{aligned} \mu^* = \max \quad & \text{trace}QX \\ \text{s.t.} \quad & \text{diag}(X) = e \\ & \text{rank}(X) = 1 \\ & X \succeq 0. \end{aligned}$$

We denote the cut polytope as the convex hull of the matrix X for all $v \in \{-1, 1\}^n$ by C_n and metric polytope by M_n:

$$\begin{aligned} C_n &= \text{Conv}\{X : X = vv^T, v \in \{-1, 1\}^n\} \subseteq S^n. \\ M_n &= \{X \in S^n : \text{diag}(X) = e, \text{ and} \\ &\quad X_{ij} + X_{ik} + X_{jk} \geq -1, X_{ij} - X_{ik} - X_{jk} \geq -1, \\ &\quad -X_{ij} + X_{ik} - X_{jk} \geq -1, -X_{ij} - X_{ik} + X_{jk} \geq -1, \\ &\quad \forall\, 1 \leq i < j < k \leq n\}. \end{aligned}$$

These are well known that $C_n \subseteq \varepsilon_n \cap M_n$ for all n, where

$$\varepsilon_n = \{X \in S^n : \mathrm{diag}(X) = e, X \succeq 0\}.$$

Finally, we define:

$$H(i, j) = \begin{cases} t(j-2) + i, & \text{if } i < j \\ t(i-2) + j, & \text{if } i > j \end{cases}.$$

3. A tight relaxation for MC problem

Let

$$H_c = \begin{pmatrix} \alpha & \mathrm{hsvec}(X)^T \\ \mathrm{hsvec}(Q) & 0 \end{pmatrix}, \quad \alpha = \sum_{i=1}^{n} Q_{ii}.$$

We easily obtain:

Lemma 3.1. $\forall X \in S^n$ *and* $\mathrm{diag}(X) = e$,

$$Z = \begin{pmatrix} 1 & \mathrm{hsvec}(Q)^T \\ \mathrm{hsvec}(X) & \bar{Z} \end{pmatrix} \in S^{t(n-1)+1},$$

then $\mathrm{trace} QX = \mathrm{trace} H_c Z$.

Now, let us consider the rank-one matrix $X = vv^T$, $v \in \{-1, 1\}^n$. For $x = \mathrm{hsvec}(X)$, $Z = \{ \begin{smallmatrix} 1 & x^T \\ x & xx^T \end{smallmatrix} \}$. We have

$$\begin{aligned} Z_{0,H(i,j)} &= x_{H(i,j)} = v_i v_j \\ Z_{H(i,k),H(m,j)} &= x_{H(i,k)} x_{H(m,j)} = v_i v_k v_m v_j \end{aligned}$$

Where, the entry $Z_{0,H(i,j)}$ is in the first row of Z. If $k = m$, then

$$\begin{aligned} Z_{H(i,k),H(k,j)} &= v_i v_k^2 v_j = v_i v_j = Z_{0,H(i,j)} \\ Z_{H(i,k),H(k,i)} &= v_i^2 v_k^2 = 1. \end{aligned}$$

This discussion leads us to define the SDP2

$$(\mathbf{SDP2}) \quad \begin{aligned} \mu_2^* = \max \quad & \mathrm{trace}\, H_c Z \\ \text{s.t.} \quad & \mathrm{diag}(Z) = e \\ Z_{H(i,k),H(k,j)} = Z_{0,H(i,j)}, \quad & \forall k \neq i, \forall k \neq j, \forall 1 \leq i < j \leq n \\ Z \succeq 0, Z \in S^{t(n-1)+1}. & \end{aligned}$$

Note that SDP2 has $(n-1)\cdot t(n-1)+1$ equality constraints. It is easy to see that the unit matrix $I_{t(n-1)+1}$ is feasible point of SDP2, hence SDP2 satisfies the Slater constraint qualification (dual strict feasibility is easy to be obtained).

Define

$$\Phi_n = \left\{X \in S^n : X = \text{hsMat}(Z_{0,1:t(n-1)}),\, Z \text{ feasible for SDP2}\right\}.$$

Since the feasible set of SDP2 is convex and compact, and Φ_n is the image of that feasible set of SDP2, it is easy to know that Φ_n is also convex and compact.

Consider any extreme point of C_n, $X = vv^T$, $v \in \{-1, 1\}^n$.

Let $x = \text{hsvec}(X)$ or $X = \text{hsMat}(x)$, $Z = \binom{1}{x}(1 \;\; x^T) = \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix}$ is feasible for SDP2. Hence we have:

Lemma 3.2. $C_n \subseteq \Phi_n$

Suppose $X \in \Phi_n$, we have $X = \text{hsMat}(Z_{0,1:t(n-1)})$, Z feasible for SDP2. Thus $\text{diag}(X) = e$, $X_{i,j} = Z_{0,H(i,j)}$, $1 \le i < j \le n$. Since, $Z_{H(i,k),H(k,j)} = Z_{0,H(i,j)}$, $\forall k \ne i, k \ne j$, $1 \le i < j \le n$. We define Y is a $n \times n$ principal submatrix consisting of 0, $H(1,2)$, $H(1,3), \ldots, H(1,n)$ rows of Z and 0, $H(1,2)$, $H(1,3), \ldots, H(1,n)$ columns of Z. Obviously, $X = Y$. Since $Z \succeq 0$, $X \succeq 0$.

So we prove the following lemma:

Lemma 3.3. $\Phi_n \subseteq \varepsilon_n$.

We now claim that

Theorem 3.1. $C_n \subseteq \Phi_n \subseteq \varepsilon_n \cap M_n$.

Proof: Suppose $X \in \Phi_n$, then $X = \text{hsMat}(Z_{0,1:t(n-1)})$ for some Z feasible for SDP2. Given $\forall i, j, k$ such that $1 \le i < j < k \le n$, let $Z_{i,j,k}$ denote the 4×4 principal submatrix of Z corresponding to the indices 0, $H(i,j)$, $H(i,k)$, $H(j,k)$. □

Let $a = X_{i,j} = Z_{0,H(i,j)}$, $b = X_{i,k} = Z_{0,H(i,k)}$, $c = X_{j,k} = Z_{0,H(j,k)}$. Since $\text{diag}(Z) = e$ and $Z_{H(i,j),H(i,k)} = Z_{H(j,i),H(i,k)} = Z_{0,H(j,k)}$, $Z_{H(j,k),H(i,k)} = Z_{H(j,k),H(k,i)} = Z_{0,H(i,j)}$, $Z_{H(i,j),H(j,k)} = Z_{0,H(i,k)}$, then

$$Z_{i,j,k} = \begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{pmatrix}$$

Let $e_1^T = (-1, 1, 1, -1)$, $e_2^T = (-1, -1, 1, 1)$, $e_3^T = (-1, 1, -1, 1)$, since $Z_{i,j,k} \succeq 0$, then

$$e^T Z_{i,j,k} e \ge 0,\;\; e_1^T Z_{i,j,k} e_1 \ge 0,\, e_2^T Z_{t,j,k} e_2 \ge 0,\, e_3^T Z_{i,j,k} e_3 \ge 0,$$

hence, $a+b+c \geq -1$, $-a-b+c \geq -1$, $a-b-c \geq -1$, $-a+b-c \geq -1$. We obtain

$$\Phi_n \subseteq M_n.$$

Therefore, by Lemmas 3.2 and 3.3, we have

$$C_n \subseteq \Phi_n \subseteq \varepsilon_n \cap M_n.$$

Furthermore, by Lemma 3.1 and Theorem 3.1, we have

Theorem 3.2. $\mu^* \leq \mu_2^* \leq \mu_1^*$.

Thus, we obtain a tight semidefinite relaxation (SDP2) of the MAX CUT problem, which improves several previous SDP relaxations in the literature.

Now, we consider some qualities of Φ_n. Obviously, when $n \leq 4$, equalities hold in Theorem 3.1. In particular, for an arbitrary graph G we have:

Corollary 1 (see Barahona and Mahjoub, 1986; Laurent et al., 1997 for details)**.** $\mu^* = \mu_2^* = \mu_1^*$ $C_n = \Phi_n = \varepsilon_n \cap M_n$ *if* G *has no* K_5*-minor.*

By two examples, we prove that the inclusions in Theorem 3.1 are strict for $n = 5$, and hence for all $n \geq 5$.

Example 1. Consider the matrix

$$X = \begin{pmatrix} 1 & -0.6435 & -0.6435 & -0.6435 & 0.9207 \\ -0.6435 & 1 & 0.2970 & 0.2970 & -0.6435 \\ -0.6435 & 0.2970 & 1 & 0.2970 & -0.6435 \\ -0.6435 & 0.2970 & 0.2970 & 1 & -0.6435 \\ 0.9207 & -0.6435 & -0.6435 & -0.6435 & 1 \end{pmatrix}$$

It is easy to check that $X \in \varepsilon_n \cap M_n$. However, using the following SDP3 we prove that there is no matrix Z feasible for SDP2 such that $\mathrm{hsMat}(Z_{0,1:t(4)}) = X$.

$$\text{(SDP3)} \quad \begin{aligned} \rho^* = \min \quad & \tfrac{1}{11}\mathrm{trace} Z \\ \text{s.t.} \quad & Z_{0,H(i,j)} = X_{i,j}, && 1 \leq i < j \leq 5 \\ & Z_{H(i,k),H(k,j)} = X_{i,j}, && \forall k \neq i, \forall k \neq j, 1 \leq i < j \leq 5 \\ & Z_{i,i} = Z_{(i+1),(i+1)}, && i = 1, \ldots, 10 \\ & Z \succeq 0, Z \in S^{11}. \end{aligned}$$

It is easy to have the following result:

Lemma 3.4. *There is a matrix* Z *feasible for SDP3 such that* $\mathrm{hsMat}(Z_{0,1:t(4)}) = X$ *if and only if* $\rho^* \leq 1$.

Using SDPpack (version 0.9 Beta Alizadeh et al., 1997), we obtain the dual objective value which is equal to $1.000293 > 1$. Hence $\rho > 1$ holds, there is no matrix Z feasible for SDP3 such that $\text{hsMat}(Z_{0,1:t(4)}) = X$. So, $X \notin \Phi_5$. And hence for all $n \geq 5$.

Example 2. Consider the matrix

$$X = \begin{pmatrix} 1 & -0.24 & -0.24 & -0.24 & -0.24 \\ -0.24 & 1 & -0.24 & -0.24 & -0.24 \\ -0.24 & -0.24 & 1 & -0.24 & -0.24 \\ -0.24 & -0.24 & -0.24 & 1 & -0.24 \\ -0.24 & -0.24 & -0.24 & -0.24 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & -0.25 & -0.25 & -0.25 & -0.25 \\ -0.25 & 1 & -0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & 1 & -0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 1 & -0.25 \\ -0.25 & -0.25 & -0.25 & -0.25 & 1 \end{pmatrix}$$

Because $\text{trace}QX = 6.2 > 6$, $X \notin C_5$. We obtain a 11×11 matrix Z^* which is feasible for SDP2 and such that $\text{hsMat}(Z^*_{0,1:t(n-1)}) = X$, thus $X \in \Phi_5$. The matrix Z^* is included in Appendix A. Using SDPpack, by SDP3, we obtain the primal objective value is equal to $0.96 < 1$. We also can prove that $X \in \Phi_5$, by Lemma 3.4.

Thus we have result is:

Corollary 2. *$C_n \subset \Phi_n \subset \varepsilon_n \cap M_n$ for all $n \geq 5$.*

4. Numerical comparison of the relaxations

The relaxations SDP1 and SDP2 were compared using the software package SDPpack. The results are summarized in Table 1. The value ρ equals the value of the optimal cut divided by the bound, and R.E. denotes the relative error with respect to the optimal cut.

$$A(G_1) = \begin{pmatrix} 0 & 1.52 & 1.52 & 1.52 & 0.16 \\ 1.52 & 0 & 1.60 & 1.60 & 1.52 \\ 1.52 & 1.60 & 0 & 1.60 & 1.52 \\ 1.52 & 1.60 & 1.60 & 0 & 1.52 \\ 0.16 & 1.52 & 1.52 & 1.52 & 0 \end{pmatrix}, \quad A(G_2) = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Table 1. Numerical results for small test problems.

n	MC optimal value	SDP1 bound	SDP2 bound	Graph
5	4	4.5225 $\rho = 0.8845$ R.E.: 13.06%	4.0000 $\rho = 1.0000$ R.E.: 0%	5-cycle with unit edge weighs
5	6	6.2500 $\rho = 0.9600$ R.E.: 4.17%	6.2500 $\rho = 0.9600$ R.E.: 4.17%	K_5 with unit edge weighs
5	9.28	9.6040 $\rho = 0.9663$ R.E.: 3.49%	9.2800 $\rho = 1.0000$ R.E.: 0%	Given by $A(G_1)$ below
5	7	7.2500 $\rho = 0.9655$ R.E.: 3.57%	7.0358 $\rho = 0.9949$ R.E.: 0.51%	Given by $A(G_2)$ below
12	88	90.3919 $\rho = 0.9735$ R.E.: 2.72%	88.0000 $\rho = 1.0000$ R.E.: 9.9e-7	Given by $A(G_3)$ in Appendix B
11	10	10.7772 $\rho = 0.9279$ R.E.: 7.77%	10.0000 $\rho = 1.0000$ R.E.: 0%	11-cycle with unit edge weighs
9	–	11.0099	10.9856	Randomly generated
12	–	20.2766	20.2287	Randomly generated

Appendix A

The matrix Z^* for Example 2 is formed with the following column

Columns 1 through 7

1.0000	−0.2400	−0.2400	−0.2400	−0.2400	−0.2400	−0.2400
−0.2400	1.0000	−0.2400	−0.2400	−0.2400	−0.2400	0.3750
−0.2400	−0.2400	1.0000	−0.2400	−0.2400	0.3750	−0.2400
−0.2400	−0.2400	−0.2400	1.0000	0.3750	−0.2400	−0.2400
−0.2400	−0.2400	−0.2400	0.3750	1.0000	−0.2400	−0.2400
−0.2400	−0.2400	0.3750	−0.2400	−0.2400	1.0000	−0.2400
−0.2400	0.3750	−0.2400	−0.2400	−0.2400	−0.2400	1.0000
−0.2400	−0.2400	−0.2400	0.3750	−0.2400	0.3750	0.3750
0.2400	−0.2400	0.3750	−0.2400	0.3750	−0.2400	0.3750
−0.2400	0.3750	−0.2400	−0.2400	0.3750	0.3750	−0.2400
−0.2400	0.3750	0.3750	0.3750	−0.2400	−0.2400	−0.2400

Columns 8 through 11

−0.2400	−0.2400	−0.2400	−0.2400
−0.2400	−0.2400	0.3750	0.3750
−0.2400	0.3750	−0.2400	0.3750
0.3750	−0.2400	−0.2400	0.3750
−0.2400	0.3750	0.3750	−0.2400
0.3750	−0.2400	0.3750	−0.2400
0.3750	0.3750	−0.2400	−0.2400
1.0000	−0.2400	−0.2400	−0.2400
−0.2400	1.0000	−0.2400	−0.2400
−0.2400	−0.2400	1.0000	−0.2400
−0.2400	−0.2400	−0.2400	1.0000

The eigvalue of Z^* is

$$d = [1.8550; 1.8550; 1.8550; 1.8550; 1.8550; 0.0100; 0.0100; 0.0100; 0.0100; 0.0674; 1.6176];$$

Appendix B

The matrix $A(G_3)$ for test problem with 12 vertices,

Columns 1 through 12

0	2	2	0	2	4	0	2	2	2	2	2
2	0	0	2	4	2	2	0	2	2	2	2
2	0	0	0	4	4	2	4	4	0	2	2
0	2	0	0	0	2	0	0	2	4	2	2
2	4	4	0	0	2	2	2	2	4	2	4
4	2	4	2	2	0	2	0	2	2	2	2
0	2	2	0	2	2	0	0	0	4	2	2
2	0	4	0	2	0	0	0	0	4	4	2
2	2	4	2	2	2	0	0	0	2	2	2
2	2	0	4	4	2	4	4	2	0	4	2
2	2	2	2	2	2	2	4	2	4	0	4
2	2	2	2	4	2	2	2	2	2	4	0

References

M. Anjos and H. Wolkowicz, "A strengthened SDP relaxation via a second lifting for the Max Cut problem," Technical Report Research Report CORR 95-55, University of Waterloo, Waterloo, Ontario, 1999.

M. Anjos and H. Wolkowicz, "A tight semidefinite relaxation of the cut polytope," Research Report CORR 2000-19, University of Waterloo, Waterloo, Ontario, 2000.

F. Alizadeh, J.-P. Haeberly, M.V. Nayakkankuppam, M.L. Overton, and S. Schmieta, "*SDPpack* user's guide—version 0.9 Beta," Technical Report TR 1997-737, Courant Institute of Mathematical Science, NYU, New York, NY, June 1997.

F. Barahona, M. Grotschel, M. Junger, and G. Reinelt, "An application of combinatorial optimization to statistical physics and circuit layout design," *Operations Research*, vol. 36, no. 3, pp. 493–513, 1988.

F. Barahona and A.R. Mahjoub, "On the cut polytope," *Mathematical Programming*, vol. 36, no. 1, pp. 157–173, 1986.

M.X. Goemans and D.P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problem using semidefinite programming," *Journal of the ACM*, vol. 42, no. 6, pp. 1115–1145, 1995.

M. Laurent, S. Poljak, and F. Rendl, "Connections between semidefinite relaxations of the max-cut and stable set problems," *Mathematical Programming*, vol. 77, no. 2, pp. 225–246, 1997.