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MOTZKIN'S TRANSPOSITION THEOREM, AND THE RELATED THEOREMS OF FARKAS, GORDAN AND STIEMKE

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Motzkin's thesis [6], in particular his *Transposition Theorem* (Theorems 1–2 below), was a milestone in the development of linear inequalities and related areas.

For two vectors $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_i)$ of equal dimension we denote by $\mathbf{u} \geqq \mathbf{v}$ and $\mathbf{u} > \mathbf{v}$ that the indicated inequality holds componentwise, and by $\mathbf{u} \gneqq \mathbf{v}$ the fact $\mathbf{u} \geqq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$.

Systems of linear inequalities appear in several forms; the following examples are typical:

(a) $A\mathbf{x} \leqq \mathbf{b}$ (b) $A\mathbf{x} = \mathbf{b}\ ,\ \mathbf{x} \geqq \mathbf{0}$ (c) $A\mathbf{x} \leqq \mathbf{b}\ ,\ B\mathbf{x} < \mathbf{c}$
(d) $A\mathbf{x} = \mathbf{0}\ ,\ \mathbf{x} \gneqq \mathbf{0}$ (e) $A\mathbf{x} = \mathbf{0}\ ,\ \mathbf{x} > \mathbf{0}$ (f) $A\mathbf{x} > \mathbf{0}\ ,\ B\mathbf{x} \geqq \mathbf{0}\ ,\ C\mathbf{x} = \mathbf{0}$

In each of these systems, called *primal*, the existence of solutions is characterized by means of a *dual system*, using the transposes of matrices in the primal system. Hence the name *transposition theorem*. The relation between the primal and dual systems is sometimes given as a *theorem of alternatives*, listing *alternatives*, i.e. statements P, Q satisfying $P \iff \neg Q$ ($\neg$ denotes negation), in words: *either P or Q but never both*.

Relations between the above systems: (a) and (b) are equivalent representations. Indeed, (a) and (b) can be written as

$$(A, -A, I)\begin{pmatrix}\mathbf{x}^+\\ \mathbf{x}^-\\ \mathbf{s}\end{pmatrix} = \mathbf{b}\ ,\ \begin{pmatrix}\mathbf{x}^+\\ \mathbf{x}^-\\ \mathbf{s}\end{pmatrix} \geq \mathbf{0} \quad\text{and}\quad \begin{pmatrix}A\\ -A\\ -I\end{pmatrix}\mathbf{x} \leq \begin{pmatrix}\mathbf{b}\\ -\mathbf{b}\\ \mathbf{0}\end{pmatrix}\ ,\ \text{respectively.}$$

The remaining systems involve strict inequalities or nontrivial solutions. For example, (d) and (e) concern the existence of nontrivial solutions and positive solutions, respectively, for the positively homogeneous system

$$A\mathbf{x} = \mathbf{0}\ ,\ \mathbf{x} \geqq \mathbf{0}\ .$$

Taking $B = O$ and $\mathbf{c} > \mathbf{0}$ in (c) gives (a). Therefore, (a) and (b) are special cases of (c). Similarly, the systems (d) and (e) are special cases of (f), which itself is a special case of (c) with $\mathbf{b} = \mathbf{0}\ ,\ \mathbf{c} = \mathbf{0}$. In fact, every system of linear inequalities can be written as (c).

The following two versions of *Motzkin's Transposition Theorem*, [6], concern systems (c) and (f).

Theorem 1 (Solvability of (c))**.** Given matrices A, B and vectors $\mathbf{b}, \mathbf{c}$, the following are equivalent:

(c1) the system $A\mathbf{x} \leqq \mathbf{b}\ ,\ B\mathbf{x} < \mathbf{c}$ has a solution $\mathbf{x}$
(c2) for all vectors $\mathbf{y} \geqq \mathbf{0}\ ,\ \mathbf{z} \geqq \mathbf{0}$,
$A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{0} \implies \mathbf{b}^T\mathbf{y} + \mathbf{c}^T\mathbf{z} \geq 0$ and
$A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{0}\ ,\ \mathbf{z} \neq \mathbf{0} \implies \mathbf{b}^T\mathbf{y} + \mathbf{c}^T\mathbf{z} > 0.$ □

Theorem 2 (Solvability of (f))**.** Let A, B, C be given matrices, with A nonvacuous. Then the following are alternatives:

(f1) $A\mathbf{x} > \mathbf{0}\ ,\ B\mathbf{x} \geqq \mathbf{0}\ ,\ C\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x}$,
(f2) $A^T\mathbf{y}_1 + B^T\mathbf{y}_2 + C^T\mathbf{y}_3 = \mathbf{0}\ ,\ \mathbf{y}_1 \gneqq \mathbf{0}\ ,\ \mathbf{y}_2 \geqq \mathbf{0}$ has solutions $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, □

Special cases of Motzkin's Theorem include the following four theorems. First, the celebrated Farkas' Theorem, [2].

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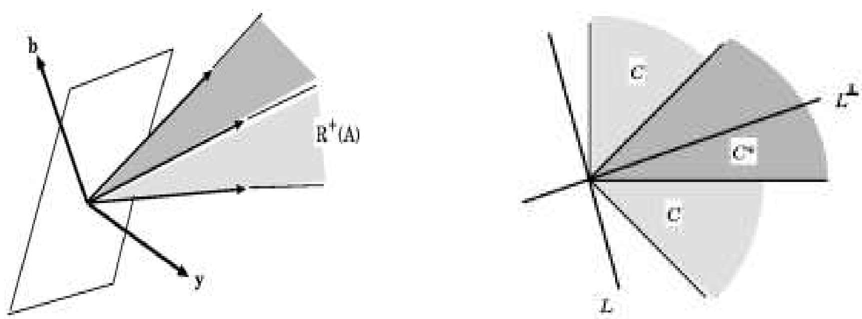
Theorem 3 (Farkas' Theorem for system (a))**.** Given a matrix A and a vector $\mathbf{b}$, the following are equivalent:
(a1) the system $A\,\mathbf{x} \leqq \mathbf{b}$ has a solution $\mathbf{x}$
(a2) $A^T\mathbf{y} = \mathbf{0}\ ,\ \mathbf{y} \geqq \mathbf{0} \implies \mathbf{b}^T\mathbf{y} \geq 0.$ □

Theorem 4 (Farkas' Theorem for system (b))**.** Given a matrix A and a vector $\mathbf{b}$, the following are equivalent:
(b1) the system $A\,\mathbf{x} = \mathbf{b}\ ,\ \mathbf{x} \geqq \mathbf{0}$ has a solution $\mathbf{x}$
(b2) $A^T\mathbf{y} \geqq \mathbf{0} \implies \mathbf{b}^T\mathbf{y} \geq 0.$ □

The positively homogeneous systems (d) and (e) are covered by the following two theorems.

Theorem 5 ([3])**.** Given a matrix A, the following are alternatives:
(d1) $A\mathbf{x} = \mathbf{0}\ ,\ \mathbf{x} \gneqq \mathbf{0}$ has a solution $\mathbf{x}$,
(d2) $A^T\,\mathbf{y} > \mathbf{0}$ has a solution $\mathbf{y}$ □

Theorem 6 ([10])**.** Given a matrix A, the following are alternatives:
(e1) $A\mathbf{x} = \mathbf{0}\ ,\ \mathbf{x} > \mathbf{0}$ has a solution $\mathbf{x}$,
(e2) $A^T\,\mathbf{y} \gneqq \mathbf{0}$ has a solution $\mathbf{y}$, □



(a) A hyperplane with normal $\mathbf{y}$ separating $\mathbf{b}$ and $\mathbb{R}^+(A)$

(b) Illustration of the alternatives (6) $L \cap C = \{\mathbf{0}\}, L^\perp \cap \operatorname{int} C^* \neq \emptyset$

FIGURE 1. Illustrations

The above results are *Separation Theorems*, or statements about the existence of hyperplanes separating certain disjoint convex sets. First, some terminology. A set $P \subset \mathbb{R}^n$ is *polyhedral* (and necessarily convex) if it is the intersection of finitely many closed halfspaces, say

$$P := \{\mathbf{x} : \ B\mathbf{x} \leqq \mathbf{b}\}\ ,\ \text{for some matrix } B \text{ and vector } \mathbf{b}\ . \tag{1}$$

A *finitely generated cone* is the set of nonnegative linear combinations of finitely many vectors (*generators*), for example, the cone generated by the columns of a matrix A,

$$\mathbb{R}^+(A) := \{A\mathbf{x} : \ \mathbf{x} \geqq \mathbf{0}\}\ . \tag{2}$$

The *dual* (or *polar*) S^* of a nonempty set $S \subset \mathbb{R}^n$ is defined as

$$S^* := \{\mathbf{y} : \ \mathbf{s} \in S \Longrightarrow \mathbf{y}^T\mathbf{s} \geq 0\}\ , \tag{3}$$

a closed convex cone. In particular,

$$\left(\mathbb{R}^+(A)\right)^* \ = \ \{\mathbf{y} : \ A^T\mathbf{y} \geqq \mathbf{0}\}\ ,\ \text{a polyhedral cone}\ . \tag{4}$$

Theorem 4(b1) states that the vector $\mathbf{b}$ is in the cone $\mathbb{R}^+(A)$. The equivalent statement (b2) says that $\mathbf{b}$ cannot be separated from $\mathbb{R}^+(A)$ by a hyperplane: such a separating hyperplane would have a normal $\mathbf{y}$, see Fig. 1(a), satisfying

$$\mathbf{b}^T\mathbf{y} < 0 \text{ and } \mathbf{v}^T\mathbf{y} \geq 0 \text{ for all } \mathbf{v} \in \mathbb{R}^+(A)\ ,$$

which by (4) is a negation of (b2). Farkas' Theorem 4 states that for any matrix A,

$$\mathbb{R}^+(A) = \left(\mathbb{R}^+(A)\right)^{**} . \tag{5}$$

In general, a set $C \subset \mathbb{R}^n$ is a closed convex cone if and only if $C = C^{**}$. Theorem 4 also implies that a cone in $\mathbb{R}^n$ is polyhedral if and only if it is finitely generated (the *Farkas-Minkowski-Weyl Theorem*, [9, Corollary 7.1a]. More generally, a set $S \subset \mathbb{R}^n$ is polyhedral if and only if it is the sum of a finitely generated cone and the convex hull of finitely many points (the *Minkowski–Steinitz–Weyl Theorem*, [9, Corollary 7.1b].

Theorems 5–6 can be interpreted as geometric statements about intersections $C \cap L$, of a closed convex cone C and a subspace L in $\mathbb{R}^n$. Let $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$. Thus:

Theorem 5(d1): $\mathbb{R}^n_+ \cap N(A) \neq \mathbf{0}$, where $N(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ the *nullspace* of A.

Theorem 6(e1): $\mathrm{int}(\mathbb{R}^n_+) \cap N(A) \neq \emptyset$, where $\mathrm{int}(\mathbb{R}^n_+) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$.

In each case, the dual system uses the intersection $C^* \cap L^\perp$ where $L^\perp$ is the *orthogonal complement* of L. For example, the statements ($\exists$ denotes *there exists*, int denotes *interior*)

$$\exists \mathbf{0} \neq \mathbf{x} \in C \cap L \quad \text{and} \quad \exists \mathbf{y} \in (\mathrm{int}\, C^*) \cap L^\perp \tag{6}$$

are mutually exclusive, see Fig. 1(b), for otherwise

$$\mathbf{x}^T\mathbf{y} \begin{cases} = 0 & \text{since } \mathbf{x} \perp \mathbf{y} \\ > 0 & \text{since } \mathbf{0} \neq \mathbf{x} \in C \text{ , } \mathbf{y} \in \mathrm{int}\, C^* \end{cases}$$

To make the statements in (6) alternatives, we need to show that one of them occurs, the hard part of the proof. Returning to Theorems 5–6, recall that $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ and $N(A)^\perp = R(A^T)$. Then:

Gordan's Theorem: (d1) $\mathbb{R}^n_+ \cap N(A) \neq \mathbf{0}$ and (d2) $\mathrm{int}(\mathbb{R}^n_+) \cap R(A^T)$ are alternatives.

Stiemke's Theorem: (e1) $\mathrm{int}(\mathbb{R}^n_+) \cap N(A) \neq \emptyset$ and (e2) $\mathbb{R}^n_+ \cap R(A^T) \neq \mathbf{0}$ are alternatives.

Further readings:
History: [9, pp. 209–228].
Theorems of alternatives: [5, pp. 27–37].
Generalizations: [11],[1],[8, §§ 21–22, specially Theorems 21.1,22.6].
Applications: [7],[5, p. 100].

Biographical note: Theodore S. Motzkin (Basel, 1900–Los Angeles, 1970) made important contributions to linear inequalities and polyhedral combinatorics. His name is associated with the *Motzkin transposition theorem*, *Motzkin numbers*, *Motzkin paths*, *Fourier-Motzkin elimination method* and its dual, the *double description method*. His father, Leo Motzkin (1867–1933), who studied Mathematics and Sociology at the University of Berlin, was a Zionist politician and cultural leader. The Israeli city of Kiryat Motzkin is named after him.

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