DOUBLY STOCHASTIC MATRICES AND COMPLEX VECTOR SPACES.*

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A doubly stochastic (d. s.) matrix is a matrix P such that $P_{ij} \geqq 0$, $\sum_i P_{ij} = \sum_j P_{ij} = 1$ for all i and j. A. Horn has proved

THEOREM 1. *If $y = Px$, where x, y are complex n-vectors, and P is a* d. s. *matrix, and $c_1, c_2, \cdots, c_n$ are any complex numbers, then $\sum_{i=1}^n c_i y_i$ lies in the convex hull of all the points $\sum_{i=1}^n c_i x_{\alpha i}$, $\alpha \varepsilon R^n$, where R^n is the set of all the permutations of $(1, \cdots, n)$*

and conjectured the truth of

THEOREM 2. *If x, y are complex n-vectors and $c_1, c_2, \cdots, c_n$ are any complex numbers imply that $\sum_{i=1}^n c_i y_i$ lies in the convex hull of the vectors $\sum_{i=1}^n c_i x_{\alpha i}$, $\alpha \varepsilon R^n$, then $y = Px$ where P is a* d. s. *matrix.*

In what follows Theorem 2 is established.

Let E be complex n-space. Let η represent the general complex linear functional on E $(\eta \varepsilon E^*)$ and the value of η for some $x \varepsilon E$ is represented by (η, x). If we consider E as real $2n$-space, then each real linear functional ρ on E has the property that for some $\eta \varepsilon E^*$ $(\rho, x) = R(\eta, x)$, where $R(\eta, x)$ is the real part of (η, x).

LEMMA 1. *Let X be a compact convex set in E. Suppose that for each $\eta \varepsilon E^*$ $(\eta, y) \varepsilon (\eta, X) = \{(\eta, z) : z \varepsilon X\}$. Then $y \varepsilon X$.*

Proof. Since $(\eta, y) \varepsilon (\eta, X)$, it follows $R(\eta, y) \varepsilon R(\eta, X)$. But then from a standard separation theorem ([2], p. 47) it follows that $y \varepsilon X$.

If Lemma 1 is applied to the case where X is the convex hull of the vectors which are derived from x by taking all permutations of the components of x relative to a fixed complex coordinate system, then $y \varepsilon X$ for y satisfying the hypothesis of Theorem 2. Now note

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LEMMA 2. *Let G be a finite collection $\{G_1, G_2, \cdots, G_n\}$ of linear transformations $E \rightarrow E$. Let $x \varepsilon E$. Denote by $K(G)$ the convex hull of G. Denote by $K(x)$ the convex hull of $Gx = \{G_1x, G_2x, \cdots, G_nx\}$. If $y \varepsilon K(x)$, then $y = Dx$ where $D \varepsilon H(G)$.*

Proof. Since $y \varepsilon K(x)$ it follows that $y = \sum_{i=1}^{n} w_i(G_ix)$, with $w_i \geqq 0$, $\sum w_i = 1$, and so $y = Dx$ with $D = \sum w_iG_i \varepsilon H(G)$. (There are extensions to the case where G is not finite but $K(x)$ is compact; since such results are not needed in the sequel they are not presented here.) The application of Lemma 2 to $y \varepsilon X$ implies that $y = Dx$ with x, y elements of the complex vector space E and D an n by n d. s. matrix (since D is a convex combination of permutation matrices). This establishes Theorem 2.

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REFERENCES.

[1] A. Horn, "Doubly stochastic matrices and the diagonal of a rotation matrix," *American Journal of Mathematics*, vol. 76 (1954), pp. 620-630.

[2] W. Fenchel, "Convex cones, sets and functions," Princeton University Logistics Research Project, September, 1953.