

20. Lecture, 9 November 1999

20.1 Johnson noise

Believe it or not, an ordinary resistor, sitting on the table not hooked up in a circuit, has a fluctuating voltage across it. The *average* voltage is zero, of course, but since the resistor has free electrons that can be thermally agitated, small, short-lived voltage fluctuations are induced – another fluctuation which therefore has its origin in the finite charge on the electron, and in the random vibrational motion solids at finite temperature. In the following we will derive an expression for the rms voltage due to these fluctuations. Most experimenters refer to this thermal voltage fluctuation as *Johnson noise*, after J.B. Johnson, who was first to observe the effect, in 1928. Our derivation will follow that by H. Nyquist, done soon after Johnson’s experiment (some call the effect *Nyquist noise*, after this initial explanation).

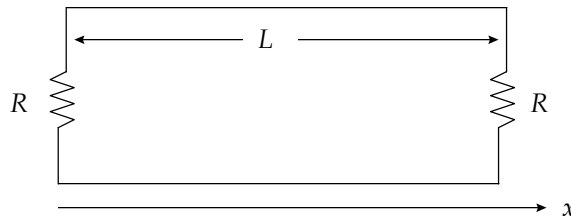


Figure 20.1: resistors and transmission line used in derivation of Johnson noise voltage.

Begin by considering *two* resistors, each having resistance R , held at temperature T . Connect their ends with two wires of length L , as in Figure 20.1. Each resistor has a thermally-fluctuating voltage which will be transmitted down the wires, appear across the other resistor, lead to a current, and appear as dissipated power, *via* $P = IV = I^2 R$. Since this involves transfer of electromagnetic energy from one resistor to the other, it is appropriate to think of *electromagnetic waves* as the agents of energy transfer. It turns out that we can find the fluctuating voltage by setting the power in waves equal to the dissipated power.

The waves traveling down the wires, in turn, can be viewed as *light* in one dimension: the pair of wires is a *transmission line*, a particularly simple form of a waveguide. Suppose the wave speed (phase “velocity”) of the pair of wires is c' (in general, different from the speed $c = 2.9979 \times 10^{10}$ cm s⁻¹ that light has in a vacuum); then a traveling wave propagating to the right has the form

$$V_R = V_{R0} e^{i(\kappa x - \omega t)}, \quad (20.1)$$

and one traveling to the left,

$$V_L = V_{L0} e^{i(-\kappa x - \omega t)}, \quad (20.2)$$

where $\kappa = \omega / c'$. Instead of dealing directly with the traveling waves, we will find it convenient to consider the *normal modes*, or *standing waves*, first, which can be made of counter-propagating traveling waves of unknown but equal amplitude V_0 :

$$\begin{aligned} V(x,t) &= V_0 e^{i(\kappa x - \omega t)} + V_0 e^{i(-\kappa x - \omega t)} \\ &= 2V_0 e^{-2\pi i f t} \cos \kappa x \end{aligned} \quad (20.3)$$

where $f = \omega / 2\pi$. These are *standing* waves only if an integer number of half-wavelengths fit between the ends:

$$\begin{aligned} L &= m \frac{\lambda}{2} \quad (m = 0, 1, 2, \dots), \text{ or} \\ \kappa &= \frac{m\pi}{L} \end{aligned} \quad (20.4)$$

In terms of the wavenumber κ (or, if you like, "in κ -space") the interval between normal modes is $(m+1)\pi/L - m\pi/L = \pi/L$. Thus if $L \gg \lambda$, then the number of modes with wavenumbers between κ and $\kappa+d\kappa$, which we can write as $N_\kappa d\kappa$, is

$$N_\kappa d\kappa = \frac{L}{\pi} d\kappa \quad (20.5)$$

Since the frequency is $f = c' / \lambda = c' \kappa / 2\pi$, and an infinitesimal frequency interval is $df = c' d\kappa / 2\pi$, the number of modes with frequency between f and $f+df$, which is the same as the number of modes with wavenumbers between κ and $\kappa+d\kappa$, is

$$N_f df = \frac{2L}{c'} df \quad (20.6)$$

We could make N_f larger or smaller by making L larger or smaller; let us define the number of modes per unit length per unit frequency interval, ρ_f , which is independent of length:

$$\rho_f = \frac{N_f df}{L df} = \frac{2}{c'} \quad (20.7)$$

Jargon: frequency interval is usually called the *bandwidth* (df , here), and ρ_f is the *one-dimensional density of modes per unit bandwidth*.

How are the modes excited? Where do they get their energy? We can think of the thermal fluctuations as adding energy to them in the form of *photons* (a large number of them, if we want a good sinusoidal wave); then, for a mode with N photons, the energy is

$$E_N = Nhf \quad (20.8)$$

where h is Planck's constant. Now, the chance that a given energy state (like E_N) is populated in *thermal equilibrium* at temperature T is given by the Boltzmann factor:

$$\begin{aligned} p(E_N) &\propto e^{-E_N/kT} \\ &\propto e^{-Nhf/kT} \end{aligned} \quad (20.9)$$

or, since the probabilities of population of all possible energy states have to add up to unity, and $0 < N < \infty$,

$$p(N) = \frac{e^{-Nhf/kT}}{\sum_{N=0}^{\infty} e^{-Nhf/kT}}. \quad (20.10)$$

where k is Boltzmann's constant. Recall the infinite sum

$$\sum_{i=0}^{\infty} y^i = \frac{1}{(1-y)} \quad (|y| < 1), \quad (20.11)$$

and note that $e^{-hf/kT} < 1$ since f , T , h and k are all positive; then we get

$$p(N) = (1 - e^{-hf/kT}) e^{-Nhf/kT}. \quad (20.12)$$

This result is called the *photon probability distribution*, and is the same as the Bose-Einstein probability distribution in the limit of zero chemical potential. From Equation 20.12 we can compute the average number of photons, or *mode occupation number*, \bar{N} , in the normal mode with frequency f :

$$\bar{N} = \sum_{N=0}^{\infty} Np(N) = (1 - e^{-x}) \sum_{N=0}^{\infty} Ne^{-Nx}, \quad (20.13)$$

where $x \equiv hf/kT$. Note that

$$Ne^{-Nx} = -\frac{\partial}{\partial x} (e^{-Nx}), \quad (20.14)$$

with which we can turn Equation 20.13 into a familiar sum:

$$\begin{aligned} \bar{N} &= (1 - e^{-x}) \sum_{N=0}^{\infty} \left(-\frac{\partial}{\partial x} e^{-Nx} \right) = -(1 - e^{-x}) \frac{d}{dx} \sum_{N=0}^{\infty} e^{-Nx} \\ &= -(1 - e^{-x}) \frac{d}{dx} \frac{1}{1 - e^{-x}} = (1 - e^{-x}) \frac{e^{-x}}{(1 - e^{-x})^2} = \frac{1}{e^x - 1} \\ &= \frac{1}{e^{hf/kT} - 1}. \end{aligned} \quad (20.15)$$

Now we can write the average energy per unit length and bandwidth; it is

$$\begin{aligned} \bar{u}_f &= \frac{\text{number of modes}}{\text{length and bandwidth}} \cdot \frac{\text{number of photons}}{\text{mode}} \cdot \frac{\text{energy}}{\text{photon}} \\ &= \rho_f \bar{N} hf = \frac{2}{c'} \frac{hf}{e^{hf/kT} - 1} \end{aligned} \quad (20.16)$$

for the normal mode with frequency f . This mode consists of counter-propagating travelling waves, each of which carries half of the energy, so the *power per unit bandwidth* emitted by either resistor is *half* the mode's energy per unit length and bandwidth, times the *length* a wave covers *per unit time* (i.e. c'), which is independent of L and c' :

$$\bar{P}_f = \frac{1}{2} u_f \cdot c' = \frac{hf}{e^{hf/kT} - 1} \quad (20.17)$$

Thus each resistor emits a total average power \bar{P} in the mode at frequency f , within a small bandwidth $\Delta f \ll f$, given by

$$\bar{P} = \bar{P}_f \Delta f = \frac{hf\Delta f}{e^{hf/kT} - 1} \quad (20.18)$$

Now we can deal with the unknown wave amplitudes in Equation 20.3. If the power given by Equation 20.18 corresponds to a voltage wave with amplitude V_0 , then a current with amplitude

$$I_0 = V_0 / 2R \quad (20.19)$$

is driven through the circuit ($2R$, because of course it flows through *both* resistors). The peak power is thus

$$P_0 = I_0^2 R = V_0^2 / 4R \quad (20.20)$$

or

$$V_0^2 = 4RP_0 \quad (20.21)$$

But the power is actually *fluctuating*, since it is driven by the power in the transmission line's standing waves, so the mean-square voltage across each resistor is

$$\overline{V^2} = 4R\bar{P} = \frac{4Rhf\Delta f}{e^{hf/kT} - 1} \quad (20.22)$$

There aren't any sources of voltage in the circuit in Figure 20.1, so $\bar{V} = 0$, and

$$\begin{aligned} \overline{(\Delta V)^2} &= \overline{V^2} - \bar{V}^2 = \overline{V^2} \\ &= \frac{4Rhf\Delta f}{e^{hf/kT} - 1} \end{aligned} \quad (20.23)$$

This is the most general form for the mean-square Johnson noise voltage fluctuations. Since we speak of low-frequency observables such as voltages and currents (rather than light, really), we usually specialize to the case $hf \ll kT$ (called the *Rayleigh-Jeans* limit), and expand the exponential in the denominator of Equation 20.21 in a power series, keeping only through first order:

$$\begin{aligned} e^{hf/kT} &\cong 1 + \frac{hf}{kT} \quad , \\ \text{or} \quad \frac{1}{e^{hf/kT} - 1} &\cong \frac{1}{1 + \frac{hf}{kT} - 1} = \frac{kT}{hf} \quad , \end{aligned} \quad (20.24)$$

so that average power per mode emitted by each resistor (Equation 20.18) is

$$\bar{P} = kT\Delta f \quad , \quad (20.25)$$

and the Johnson noise mean square and rms voltages become

$$\begin{aligned} V_N^2 &\equiv \overline{(\Delta V)^2} = 4RkT\Delta f \\ V_N &\equiv (\Delta V)_{\text{rms}} = \sqrt{4RkT\Delta f} \quad . \end{aligned} \quad (20.26)$$

The noise currents corresponding to these noise voltages can be obtained by noting Equation 20.19:

$$\begin{aligned} I_N^2 &= \frac{V_N^2}{4R^2} = \frac{kT\Delta f}{R} \quad , \\ I_N &= \frac{V_N}{2R} = \sqrt{\frac{kT\Delta f}{R}} \quad . \end{aligned} \quad (20.27)$$

Now we know two forms of noise in currents made up of carriers with quantized charge: shot noise, for which the mean square current depends upon the average current in the circuit (Equation 19.20):

$$\begin{aligned} I_N^2 &= 2qI\Delta f \\ I_N &= \sqrt{2qI\Delta f} \quad , \end{aligned} \quad (20.28)$$

and Johnson noise, which doesn't.

20.2 Johnson noise and blackbody radiation in three dimensions

It may occur to you that the reasoning leading up to Equations 20.18 and 20.25 is actually a one-dimensional derivation of the power emitted by a black body in thermal equilibrium at temperature T . Let's compare these results to blackbody radiation under the restrictions that it emits in a single polarization, and in a *single mode*; in free space, that's a diffraction-limited beam:

$$A\Omega = \lambda^2 = \frac{c^2}{\nu^2} \quad (\text{FWHM beam}).^1 \quad (20.29)$$

The blackbody power is

$$\begin{aligned} P &= \frac{1}{2} B_\nu(T) \Delta\nu A\Omega = \frac{h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \Delta\nu \frac{c^2}{\nu^2} \\ &= \frac{h\nu\Delta\nu}{e^{h\nu/kT} - 1} \cong kT\Delta\nu \quad \text{in the Rayleigh - Jeans limit,} \end{aligned} \quad (20.30)$$

¹ Generally we will use f for frequencies of currents in circuit (that is, very low frequencies) and ν for frequencies of light propagating in free space (usually rather high frequencies).

just like the single-standing-wave-mode power in the one-dimensional case. This means we can actually replace the extra resistor, and the two wires, with a surrounding blackbody, and still get Johnson noise in the resistor. This is the real explanation for the Johnson noise in the isolated resistor, laying there on the table. Since the resistor has a constant temperature T , it absorbs as much power as it emits. Both the emitted and absorbed power per mode are equal to $kT\Delta f$.

20.3 Poissonian and Gaussian statistics

The photon probability distribution, Equation 20.12, includes the Poisson distribution as one of its limiting cases. This is perhaps best illustrated by comparing the rms fluctuation from the mean. We have already calculated \bar{N} for the photon distribution (Equation 20.15), so all we need to calculate for such a comparison is $\overline{N^2}$. Using again the substitution $x \equiv hf / kT$,

$$\begin{aligned} \overline{N^2} &= (1-e^{-x}) \sum_{N=0}^{\infty} N^2 e^{-Nx} = (1-e^{-x}) \sum_{N=0}^{\infty} \frac{\partial^2}{\partial x^2} e^{-Nx} = (1-e^{-x}) \frac{d^2}{dx^2} \sum_{N=0}^{\infty} e^{-Nx} \\ &= (1-e^{-x}) \frac{d^2}{dx^2} \frac{1}{1-e^{-x}} = -(1-e^{-x}) \frac{d}{dx} \frac{e^{-x}}{(1-e^{-x})^2} \\ &= (1-e^{-x}) \left[\frac{e^{-x}}{(1-e^{-x})^2} + 2e^{-x} \frac{e^{-x}}{(1-e^{-x})^3} \right] \\ &= \frac{e^{-x} - e^{-2x} + 2e^{-2x}}{(1-e^{-x})^2} \left(\frac{e^{2x}}{e^{2x}} \right) = \frac{e^x + 1}{(e^x - 1)^2} . \end{aligned} \tag{20.31}$$

But $\bar{N} = 1/(e^x - 1)$, so

$$e^x = 1 + \frac{1}{\bar{N}} ; \tag{20.32}$$

we can use this to express $\overline{N^2}$ in terms of \bar{N} , a tactic we found useful for the moments of the Poisson distribution. Substituting Equation 20.32 into 20.31, we get

$$\overline{N^2} = \frac{2 + \frac{1}{\bar{N}}}{\left(\frac{1}{\bar{N}}\right)^2} = 2\bar{N}^2 + \bar{N} . \tag{20.33}$$

Thus the variance in N is

$$(\Delta N)^2 = \overline{N^2} - \bar{N}^2 = \bar{N}^2 + \bar{N} = \bar{N}(\bar{N} + 1) , \tag{20.34}$$

or

$$(\Delta N)_{\text{rms}} = \sqrt{\bar{N}(\bar{N} + 1)} . \tag{20.35}$$

It is worth looking at some of these results in two opposite limits. In the limit $hf \gg kT$, usually called the *Wien limit*, the mean occupation number \bar{N} is

$$\bar{N} = \frac{1}{e^{hf/kT} - 1} \cong e^{-hf/kT} \ll 1, \quad (20.36)$$

just like Poissonian statistics, and

$$(\Delta N)_{\text{rms}} \cong \sqrt{\bar{N}}, \quad (20.37)$$

also just like Poissonian statistics; at frequencies high enough that $\bar{N} \ll 1$, photons obey Poissonian statistics. In the Rayleigh-Jeans limit, $hf \ll kT$, we have already seen (Equation 20.24) that

$$\bar{N} = \frac{1}{e^{hf/kT} - 1} \cong \frac{kT}{hf} \gg 1. \quad (20.38)$$

Thus the rms deviation of N from the mean is approximately

$$(\Delta N)_{\text{rms}} \cong \sqrt{\bar{N}(\bar{N})} = \bar{N}. \quad (20.39)$$

This limit ($\bar{N} \gg 1$) is usually called that of *Gaussian statistics*. Thus, using the rms deviation as a measure of uncertainty, the value of a Poisson-distributed variable is $\bar{N} \pm \sqrt{\bar{N}}$, and that of a Gaussian random variable is $\bar{N} \pm \bar{N}$. The *relative* fluctuations $(\Delta N)_{\text{rms}} / \bar{N}$ are much larger for Gaussian statistics, a phenomenon known as *bunching*.