4. HOWARD, R. A., *Dynamic Programming and Markov Processes*, Wiley, 1960.
5. MACQUEEN, J. B., "A Test for Suboptimal Actions in Markovian Decision Problems," *Operations Research*, 15, 1967, pp. 559–561.
6. MORTON, T. E., "On the Asymptotic Convergence Rate of Cost Differences for Markovian Decision Processes," *Operations Research*, 19, 1971, pp. 244–248.
7. ——— AND WECKER, W. E., "Discounting, Ergodicity and Convergence of Markov Decision Process," Management Sciences Report No. 357, Graduate School of Industrial Administration, Carnegie-Mellon University, January 1975.
8. ODONI, A. R., "On Finding the Maximal Gain for Markov Decision Processes," *Operations Research*, 17, 1969, pp. 857–860.
9. SCHWEITZER, P. J., "Perturbation Theory and Markovian Decision Processes," MIT Operations Research Centre Technical Report No. 15, June 1965.

MANAGEMENT SCIENCE
Vol. 23, No. 1, September, 1976
Printed in U.S.A.

Notes*
II

AN ANALOGUE DERIVATION OF THE DUAL OF THE GENERAL FERMAT PROBLEM†

D. J. WHITE

University of Manchester

The General Fermat problem is as follows (see Kuhn [2]), where $X, \{X_i\}$ are column vectors.

P

Let there be given n distinct points $X_i = (x_i, y_i)$ in the plane and n positive weights w_i, $i = 1, 2, \ldots, n$. Furthermore, for $X = (x, y)$, let

$$d_i(X) = \left((x - x_i)^2 + (y - y_i^2)\right)^{\frac{1}{2}}, \tag{1}$$

the Euclidean distance from X to X_i, for $i = 1, 2, \ldots, n$. Then the General Fermat Problem asks for a point X that minimises

$$f(X) = \sum_i w_i d_i(X). \tag{2}$$

Kuhn gives the following dual of P:

D

Let $U_i = (u_i, v_i)$ denote n two-dimensional vectors. Then the dual to the General Fermat Problem asks for the vectors U_i which maximise

$$g(U_1, U_2, \ldots, U_n) = \sum_i U_i X_i \qquad \text{subject to} \tag{3}$$

$$\sum U_i = 0 \tag{4}$$

$$|U_i| \leqslant w_i, \text{ for } i = 1, 2, \ldots, n. \tag{5}$$

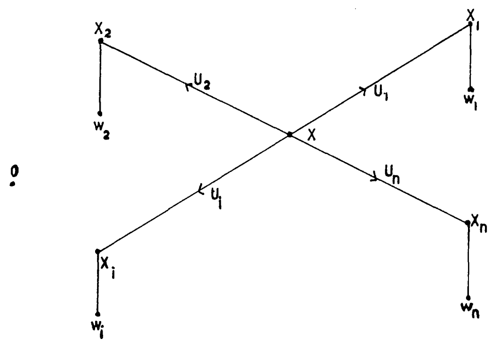
* All Notes are refereed.

† Processed by Dr. Willard I. Zangwill, Departmental Editor for Linear and Nonlinear Programming; received November 1975.

Kuhn proves some results about P and D which, in effect, show that the optimal values of P and D are equal. In White [4] it was shown how a particular duality theorem might be obtained from an analogue mechanism. This note shows how the dual problem P may also be obtained from an analogue argument.

The particular analogue to be used may be found in White [5], which is a modification of an approach by Haley [1].

The analogue consists of a smooth table with n holes through which n strings are passed with a weight proportional to w_i, $i = 1, 2, \ldots, n$, at the end of each. The holes represent the vectors X_i, $i = 1, 2, \ldots, n$. The strings are joined on the surface of the table at a point representing X. X is restricted so that it cannot be pulled through a hole. The system settles down to a position of minimal potential energy (it is easy to show that this is a global minimum). It is shown that (2) is equal to the potential energy (plus a constant) and hence the analogue solves (2) (see Figure).



Now let U_i, $i = 1, 2, \ldots, n$, denote the tension vectors relating to w_i, $i = 1, 2, \ldots, n$, respectively, measured from X, where $\{U_i\}$ are row vectors.

Condition (4) is equivalent to the statement that the net force at X is 0 in the equilibrium condition. Condition (5) is a statement that, in the equilibrium condition, either the tension is equal to w_i, at X_i, or X has been stopped by some restriction at X_i.

Let us now begin by holding X (by some external force) at some origin $0 \neq X_i$, $i = 1, 2, \ldots, n$. We will gradually displace X in small amounts, ΔX, until it is in its equilibrium position, by applying appropriate forces at X, which will diminish to 0 as we approach the equilibrium condition.

The virtual work done (see Routh [3]) for such a displacement is

$$\Delta W = \sum_i U_i \Delta(X_i - X). \quad \text{Now} \tag{6}$$

$$\sum_i U_i \Delta(X_i - X) = \Delta\Big(\sum_i U_i(X_i - X)\Big) - \sum_i \Delta U_i(X_i - X). \tag{7}$$

If T_i is the tension in string i, and if V_i is the unit vector in direction of tension, we have

$$\Delta U_i(X_i - X) = \Delta T_i V_i(X_i - X) + T_i \Delta V_i(X_i - X). \tag{8}$$

We have $\Delta V_i(X_i - X) = 0$ (ignoring second order terms, since $X_i - X$ is parallel to V_i), and, since $X_i \neq X$ implies $\Delta T_i = 0$ (since $T_i = w_i$ in all such cases), we have $\Delta T_i(X_i - X) = 0$. Hence, ignoring second order terms (6) and (7) give

$$\Delta W = \Delta\Big(\sum_i U_i(X_i - X)\Big). \tag{9}$$

Integrating from 0 to X, the virtual work done is, using (4),

$$W = \left(\sum_i U_i X_i\right)_X - \left(\sum_i U_i X_i\right)_0. \tag{10}$$

If $(PE)_X$, $(PE)_0$ are the potential energies of the system at X, 0 respectively, we have (see Routh)

$$\left(\sum_i U_i X_i\right)_X - \left(\sum_i U_i X_i\right)_0 + (PE)_X - (PE)_0 = 0. \tag{11}$$

Since $(PE)_X$ is minimal at the equilibrium point X, we see that $\sum_i U_i X_i$ is maximised at X and this establishes that D is, indeed, a dual of P, with optimal solutions corresponding to each other. The fact that we have a "global" maximum for D follows from the linearity of $g(U_1, U_2, \ldots, U_n)$, the convexity of the feasible $(U_1, U_2, \ldots, U_n)$ region.

We easily see that:

$$(PE)_X - (PE)_0 = \sum w_i d_i(X) - \sum w_i d_i(0). \quad \text{Also:} \tag{12}$$

$$(U_i X_i)_0 = (w_i V_i \cdot (-d_i V_i'))_0 = -w_i d_i(0), \quad i = 1, 2, \ldots, n. \tag{13}$$

Combining (11), (12), (13) we have, at the equilibrium point,

$$\left(\sum_i U_i \cdot X_i\right)_X = \sum_i w_i d_i(X). \tag{14}$$

This completes the duality results in that the maximal value of D is now equal to the minimal value of P.

We also see (as can be seen from Kuhn's analysis also) that:

$$(|U_i| - w_i)(X - X_i) = 0, \quad i = 1, 2, \ldots, n. \tag{15}$$

The solution procedure for P follows, as in Kuhn, once $\{U_i\}$ have been found, for then, from the analogue, $X - X_i$ is parallel to U_i, and the solution easily obtained.

References

1. Haley, K.B., "The Siting of Depots," *International Journal of Production Research*, 2, No. 1, 1962.
2. Kuhn, H. W., *On A Pair of Dual Nonlinear Programs, in Nonlinear Programming*, Ed. J. Abadie, North Holland Publishing Company, 1967, pp. 37–54.
3. Routh, E. J., *Analytical Statics*, Cambridge University Press, 1896.
4. White, D. J., "A Linear Programming Analogue, A Duality Theorem, and a Dynamic Algorithm," *Management Science*, 21, No. 1, 1974, pp. 47–59.
5. ———, *Decision Methodology*, Wiley, 1975, pp. 144–145.