Application of Linear Algebra:

Notes on Talk given to Princeton University Math Club on

Cayley-Menger Determinant and Generalized N-dimensional Pythagorean Theorem

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Abstract

This is the notes for my November 2003 talk for the Princeton University Math Club on Higher Dimensional Geometry. The focus of the talk is N-dimensional content calculation of simplexes, which leads to the proof of the Cayley-Menger Determinants. From the Cayley-Menger Determinants, I attempt to then extract a multidimensional analogue to the Pythagorean Theorem.

I. MOTIVATION: EXAMPLE IN 2D

There are two useful facts we learned about triangles in elementary geometry: Hero(n)'s Formula and the classical theorem of Pythagoras. The first of which allows us to calculate the area of an arbitrary triangle by

$$A = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}\,, \tag{1}$$

where a, b, and c are the lengths of the three legs of the triangle, and s, the semiperimeter, is defined as

$$s = \frac{1}{2}(a+b+c)\,.$$

The second tells us about the length of the hypotenuse in relation to the legs of a right triangle:

$$h^2 = l_1^2 + l_2^2\,.$$

It is useful to note that Pythagorean Theorem is a degenerate case of the law of cosines:

$$(\boldsymbol{a}+\boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{a}\cdot\boldsymbol{b}\,.$$

Naturally, we ask, are there analogues for the two facts in higher dimensions? Can we calculate the N-content of an N-simplex using only the edge lengths? Is there a criterion on N-simplices (one that is analogue to the "right"-ness of a right triangle) such that a similar relation to Pythagorean theorem holds? The answer to both of those questions, is "yes", and will be demonstrated below[1].

Finally, in an effort to save time and space, the speaker assumes familiarity on the part of the reader in basic linear algebraic concepts such as: dot products, properties of determinants, etc. Such facts, when first used, will be stated without proof. The reader is encouraged to consult texts on the subject for expositions on those facts.

[1] The French mathematician J. P. de Gua de Malves generalized the Pythagorean Theorem to three dimensions in his namesake theorem, which states that for a trirectangular tetrahedron, "the square of the area of the base (i.e. the face opposite the right trihedral angle) is equal to the sum of the squares of the areas of its other three faces."

II. Introduction: Some (applications of) Linear-Algbra

Terminology: simplex

We will be using the name *simplex* alot during this paper. As its name suggests, a simplex is the simplest N-dimensional hyper-solid that can be formed, or, it is the N-dimensional hyper-solid with the least $(N-1)$-facets. Some well known properties of simplex include:

- For an N-simplex, it has $\frac{N+1!}{(N-j)!(j+1)!}$ or $(N+1)$-Choose-$(j+1)$ j-dimensional constituents. For example, a 2-D simplex, or a triangle, has three 0-D objects (or points or vertices), and three 1-D edges. For a 3-D simplex, there are four 0-D vertices, six 1-D edges, and four 2-D facets.
- Every vertex is connected to every other vertex. And as a consequence, every vertex has N edges converging on it.

The reader is encouraged to check that it indeed is true.

Within the simplices in $\mathbb{R}^n$ we define a subset called *orthoschemes*. An orthoscheme is a simplex with vertices $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ such that $(\boldsymbol{a}_0 - \boldsymbol{a}_i) \cdot (\boldsymbol{a}_0 - \boldsymbol{a}_j) = 0$ for $0 < i < j < n+1$. In words, the relation says that there is one vertex on an orthoscheme such that any two edges emanating from that vertex are perpendicular. In two dimensions, an orthoscheme is a right triangle. In three dimensions, it is a trihedral tetrahdron.

Review of Linear Algebra: some basic facts about Determinants

Let $\boldsymbol{A}$ be an $n \times n$ matrix, we can write $\boldsymbol{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n)$ where A_i are column vectors, then the following is true:

Proposition 1: *We have some properties of of the determinant:*

1. $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^t)$
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
3. $\det(\boldsymbol{A}) = \det(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_i, \boldsymbol{A}_{i+1}, \ldots, \boldsymbol{A}_n) = -1\det(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{i+1}, \boldsymbol{A}_i, \ldots, \boldsymbol{A}_n)$
4. $\det(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_i, \ldots, \boldsymbol{A}_j, \ldots, \boldsymbol{A}_n) = \det(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_i, \ldots, \boldsymbol{A}_j + s\boldsymbol{A}_i, \ldots, \boldsymbol{A}_n)$
5. $\det(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_i + \boldsymbol{B}_i, \ldots, \boldsymbol{A}_n) = \det(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_i, \ldots, \boldsymbol{A}_n) + \det(\boldsymbol{A}_1, \ldots, \boldsymbol{B}_i, \ldots, \boldsymbol{A}_n)$

These properties are stated here without proof, readers are encouraged to look them up in any textbook on the subject.

Vector Algebra and the Volume Form

We start with an example: consider the parallelagram in $\mathbb{R}^2$ with one vertex on the origin. Then the parallelagram can be described by two vectors a and b. We also easily see that the area of the parallegram is described by $|a||b|\cos(\theta)$, or $|a \times b|$, which we can also write as the determinant

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} .$$

That fact we can generalize to

Theorem 2: *An n-dimensional parallelotope (higher dimensional equivalent of the parallelagram) with one vertex at the origin, and the edges adjacent to that vertex given by $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$, has an n-content (fancy speak for volume/area) given by the absolute value of the determinant:*

$$V = \begin{vmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,n} \end{vmatrix} \tag{2}$$

where $x_{i,j}$ denotes the j-th component of $\boldsymbol{x}_i$ in $\mathbb{R}^n$.

Proof: [2] We look at the linear transformation $\boldsymbol{A}$ from $\mathbb{R}^n$ to itself that maps $\boldsymbol{e}_i$ to $\boldsymbol{x}_i$, where $\boldsymbol{e}_i$ is the i-th orthonormal basis element. $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{pmatrix}$$

From multivariate calculus, we know that the volume is given by:

$$V = \int_{0<y_i<1} |J| dy = \int_{0<y_i<1} |\det(\boldsymbol{A})| dy = |\det(\boldsymbol{A})|$$

where J is the Jacobian and equals $\det(\boldsymbol{A})$ by definition. □

CORALLARY 3: *For an n-simplex with one vertex at the origin and the other vertices described by $\{\boldsymbol{x}_i\}_{1\le i\le n}$, its volume is $1/n!$ times the volume of the corresponding parallelotope.*

Proof: The integral becomes

$$\int_{0<\sum y_i<1} |J| dy \ ,$$

it is left as a trivial excercise (for the reader) to show that the integral produces the $n!$ factor. □

III. CAYLEY-MENGER DETERMINANT: SOME MORE LINEAR ALGEBRA

In our little discussion above, we assumed that our solid has one of the vertices on the origin. This helps with the calculation, but isn't necessarily true for everyday use. The Cayley-Menger Determinant is another way of calculating the volume of an n-simplex, which relies only on the lengths of the edges (a lot less information compared to knowing the precise locations of all vertices. Calculating the edge-lengths from the vertices' positions is trivial, but the other way around is more difficult), and thus is more applicable in real life.

THEOREM 4: ***[Cayley-Menger]** Define the $(n+1)$ by $(n+1)$ matrix $\boldsymbol{B}$ by: $B_{lm} = \|\boldsymbol{v}_l - \boldsymbol{v}_m\|_2^2$, and its companion matrix $\tilde{\boldsymbol{B}}$ by:*

$$\tilde{\boldsymbol{B}} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & B_{11} & \cdots & B_{1,n+1} \\ \vdots & \vdots & & \vdots \\ 1 & B_{n+1,1} & \cdots & B_{n+1,n+1} \end{pmatrix}$$

where $\boldsymbol{v}_\lambda$ are the $N+1$ vertices of the simplex, and $\|\boldsymbol{v}_l - \boldsymbol{v}_m\|_2$ denotes the two dimensional distance between the two vertices, i.e. B_{lm} represents the length of the edge that connects $\boldsymbol{v}_l$ and $\boldsymbol{v}_m$. Then the content of the simplex is given by:

$$V^2 = \frac{(-1)^{n+1}}{2^n (n!)^2} \det(\tilde{\boldsymbol{B}}) \tag{3}$$

Proof: By corallary (3), $V = 1/n! \cdot \det(\boldsymbol{A})$ with $\boldsymbol{A}$ as defined in the proof to theorem (2). Imagine the linear transformation defined by the translation of the point on the origin to $\boldsymbol{w}_{n+1}$, i.e. $\boldsymbol{w}_i = \boldsymbol{v}_i + \boldsymbol{w}_{n+1}$. The volume should be invariant under translation. We define the matrix $\hat{\boldsymbol{A}}$ by:

$$\hat{\boldsymbol{A}} = \begin{pmatrix} 1 & w_{1,1} & \dots & w_{1,n} \\ 1 & \vdots & & \vdots \\ 1 & w_{n,1} & \dots & w_{n,n} \\ 1 & w_{n+1,1} & \dots & w_{n+1,n} \end{pmatrix}$$

[2] Readers familiar with exterior products realize that the above is a trivial result arising from the volume form on the dual space $\mathbb{R}^{*n}$.

and it is trivial to verify $\det(\hat{\boldsymbol{A}}) = \det(\boldsymbol{A})$. Next we define $\tilde{\boldsymbol{A}}$ by:

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \|\boldsymbol{w}_1\|^2 & -2w_{1,1} & \dots & -2w_{1,n} & 1 \\ \vdots & & & & \vdots \\ \|\boldsymbol{w}_{n+1}\|^2 & -2w_{n+1,1} & \dots & -2w_{n+!,n} & 1 \end{pmatrix}$$

It is easy to see that $\det(\tilde{\boldsymbol{A}}) = (-2)^n \det(\boldsymbol{A})$. We can also define $\bar{\boldsymbol{A}}$ by:

$$\bar{\boldsymbol{A}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & w_{1,1} & \dots & w_{1,n} & \|\boldsymbol{w}_1\|^2 \\ \vdots & & & & \vdots \\ 1 & w_{n+1,1} & \dots & w_{n+!,n} & \|\boldsymbol{w}_{n+1}\|^2 \end{pmatrix}$$

which has the property $\det(\bar{\boldsymbol{A}}) = (-1)^{2n+1} \det(\boldsymbol{A})$.

If we take the product $\tilde{\boldsymbol{A}} \cdot \bar{\boldsymbol{A}}^t$, we see that simply, $(\tilde{\boldsymbol{A}}\bar{\boldsymbol{A}}^t)_{11} = 0$, and $(\tilde{\boldsymbol{A}}\bar{\boldsymbol{A}}^t)_{1i} = (\tilde{\boldsymbol{A}}\bar{\boldsymbol{A}}^t)_{i1} = 1$ for $i > 1$. The diagonal elements can be seen to evaluate to

$$\|\boldsymbol{w}_i\|^2 + \sum_j (-2) w_{i,j} w_{i,j} + \|\boldsymbol{w}_i\|^2 = 0$$

and other elements can be taken as

$$(\tilde{\boldsymbol{A}}\bar{\boldsymbol{A}}^t)_{ij} = \|\boldsymbol{w}_i\|^2 + \sum_k (-2) w_{i,k} w_{j,k} + \|\boldsymbol{w}_j\|^2 = \|\boldsymbol{w}_i - \boldsymbol{w}_j\|^2 \, .$$

From which we can conclude that $\tilde{\boldsymbol{A}}\bar{\boldsymbol{A}}^t = \tilde{\boldsymbol{B}}$ as defined in the statement in the theorem. Now, $\det(\tilde{\boldsymbol{A}}\bar{\boldsymbol{A}}^t) = \det(\tilde{\boldsymbol{A}})\det(\bar{\boldsymbol{A}})$ and that implies $\det(\tilde{\boldsymbol{B}}) = (-1)^{(n+2n+1)} 2^n (n!)^2 V^2$ which proves the theorem. □

It is interesting to note that Heron's formula, stated above, is a direct consequence of theorem (4). The reader is encouraged to calculate the determinant for $n = 2$ and check the result.

IV. Generalized Pythagorean Theorem

Next, we use the result above with the Cayley-Menger determinant to show a higher dimensional analogue of the Pythagorean Theorem.

THEOREM 5: *Given an n-orthoscheme, label the vertices $\boldsymbol{g}_0, \boldsymbol{g}_1, \dots, \boldsymbol{g}_n$ and the facets* opposite *$\boldsymbol{g}_i$ as G_i. Let V_j be the j-content function, then the following relation holds:*

$$V_{n-1}(G_0)^2 = \sum_{j=1}^{n} V_{n-1}(G_j)^2 \tag{4}$$

Proof: Since the content is invariant under translation and rotation, we can, without loss of generality, choose $\boldsymbol{g}_0$ to be the origin, and that $(\boldsymbol{g}_i, \boldsymbol{e}_j) = \|\boldsymbol{g}_i\|\delta_{ij}$ where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^n$ and $\{\boldsymbol{e}_i\}_{1\le i\le n}$ is the standard basis. Then corallary (3) gives the following:

$$(n-1)! V_{n-1}(G_j) = \begin{pmatrix} g_{1,1} & \dots & \hat{g}_{1,j} & \dots & g_{1,n} \\ \vdots & & & & \vdots \\ \hat{g}_{j,1} & & \hat{\ddots} & & \hat{g}_{j,n} \\ \vdots & & & & \vdots \\ g_{n,1} & \dots & \hat{g}_{n,j} & \dots & g_{n,n} \end{pmatrix}$$

where the hat notation means "without that term". We see immediately that, given orthogonality condition above, we have

$$V_{n-1}(G_j) = \frac{1}{(n-1)!} \det(\operatorname{diag}(\|\boldsymbol{g}_1\|, \dots, \|\hat{\boldsymbol{g}_j}\|, \dots, \|\boldsymbol{g}_n\|)) = \frac{1}{(n-1)!} \prod_{k \neq j} \|\boldsymbol{g}_k\| \,. \tag{5}$$

Combining equations (5) and (3) with (4) we see that to prove the theorem is equivalent to showing

$$\frac{(-1)^n}{2^{n-1}} \det(\tilde{\boldsymbol{B}}(G_0)) = \sum_{j=1}^{n} \prod_{k \neq j} \|\boldsymbol{g}_k\|^2 \tag{6}$$

is true.

We write out $\tilde{\boldsymbol{B}}(G_0)$:

$$\begin{pmatrix} 0 & 1 \\ 1 & \|\boldsymbol{g}_i - \boldsymbol{g}_j\|^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & (\|\boldsymbol{g}_i\|^2 + \|\boldsymbol{g}_j\|^2) \cdot (1 - \delta_{ij}) \end{pmatrix}$$

by the ordinary Pythagorean Theorem. Next *we discard all terms containing* $\boldsymbol{g}_1$ (the validity of this operation will be shown later) and call the truncated matrix $\tilde{\boldsymbol{B}}_{\hat{1}}$. Given the form of $\tilde{\boldsymbol{B}}$, we see that the term $\boldsymbol{g}_1$ only appears on the second row and the second column, and each time it appears, it is coupled to another term $\boldsymbol{g}_j$ with $j \neq 1$. By linearity of the determinant, we can separate $\det(\tilde{\boldsymbol{B}})$ into two terms: $\det(\tilde{\boldsymbol{B}}_{\hat{1}}) + \det(\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}_{\hat{1}})$. The first determinant has no terms containing $\boldsymbol{g}_1$, while the second has only terms containing $\boldsymbol{g}_1$.

(An explanation of the strategy: we seek to prove that equation (6) is an identity. To do so, a term by term evaluation of the LHS is tiring and unrewarding. So we look only at terms in the expansion of the determinant that does *not* contain a given term. Since we can cyclically permute the labels of $\boldsymbol{g}_1, \dots, \boldsymbol{g}_n$ and not change the geometry nor the orientation, we can call any of them $\boldsymbol{g}_1$. Therefore we aim at showing that all the terms on LHS that does not contain $\boldsymbol{g}_1$ is equal to the same terms on RHS. And lastly, if we can show that no term on either side contains *all* of $\boldsymbol{g}_1, \dots, \boldsymbol{g}_n$, then by the symmetry on the labelling, the theorem is proven. But the condition that no term contains all of $\boldsymbol{g}_1, \dots, \boldsymbol{g}_n$ is trivially true on RHS. On LHS, since $\tilde{B}_{11} = 0$, any non-zero term in the determinant must contain $\tilde{B}_{1i}$ for some $i > 1$. That means terms containing $\boldsymbol{g}_j$ can only come from the $n-1$ rows between 2 and $n+1$, excluding i. Therefore a term cannot contain all the vertices, which require the term to have at least n factors of the form $\boldsymbol{g}_j$.)

By row and column addition, we subtract the second column from all except the first two columns, and do the same for rows. Then we reduce the matrix to one looking like this:

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & \|\boldsymbol{g}_2\|^2 & \dots & \|\boldsymbol{g}_n\|^2 \\ 1 & \|\boldsymbol{g}_2\|^2 & -2\|\boldsymbol{g}_2\|^2 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & \|\boldsymbol{g}_n\|^2 & 0 & & -2\|\boldsymbol{g}_n\|^2 \end{pmatrix}$$

The determinant of this matrix is easily computed to be

$$(-1) * (-2)^{n-1} \prod_{j=2}^{n} \|\boldsymbol{g}_j\|^2$$

and plugging it into equation (6) we see that the leading coefficients cancel, and since the only term on the RHS that doesn't contain $\boldsymbol{g}_1$ corresponds to $V_{n-1}(G_1)$, we see the equation holds. □

V. REFERENCES

For more information on higher dimensional geometry, try *An Introduction to the Geometry of N Dimensions* by D. M. Y. Sommerville, to which I am indebted for one crucial step in the proof of the C.-M. Det. For a proof of the Gen.

Pyth. Thm. from a different approach, see *A multidimensional Law of Sines* by Igor Rivin[3]. Prof. Rivin's preprints also include a paper on *Some observations on the simplex* which is great for some more information related to this talk.

[3] You can find his preprints at the arxiv server, or through his webpage at `http://euclid.math.temple.edu/~rivin/`.