THE CONVEX HULL OF SUB-PERMUTATION MATRICES[1]

N. S. MENDELSOHN AND A. L. DULMAGE

1. **Introduction.** A combinatorial theorem [**1**; **3**] usually referred to as "the marriage problem" or "the problem of distinct representatives" has the following matrix formulation; the convex hull of the set of all n by n permutation matrices is the set of all n by n doubly stochastic matrices. In this note the above theorem is generalized.

The following notation and definitions will be used. A will represent an n by n matrix with non-negative real entries a_{ij}; S will represent the sum of all entries of A, $S=\sum_i \sum_j a_{ij}$; R_i will represent the sum of the entries in the ith row and C_j will represent the sum of the entries in the jth column; M will represent the largest row or column sum of A, $M=\max(R_i, C_j)$. Also used will be the concept of a sub-permutation matrix of rank r. By this is meant a matrix P with the following properties: (1) each entry of P is either 1 or 0; (2) each row and each column of P contains at most one 1; (3) P contains exactly r entries equal to 1. In terms of this notation the theorem quoted above becomes; a matrix A lies in the convex hull of the set of all permutation matrices if and only if $M=1$ and $S=n$. In [**2**] the authors of the present note obtain sufficient conditions in order that a matrix A with non-negative entries contain nonzero entries in the places occupied by 1 in a permutation matrix of rank r. In this note necessary and sufficient conditions are given in order that a matrix A lie in the convex hull of the sub-permutation matrices of rank $n-i$ $(i=0, 1, 2, \cdots, n-1)$.

2. THE THEOREM. *Let A be an n by n matrix whose entries are non-negative real numbers. A necessary and sufficient condition that A lie in the convex hull of all sub-permutation matrices of rank $n-i$ is that $S=n-i$ and $(n-i)/n \leqq M \leqq 1$.*

PROOF. The necessity is obtained as follows. Let $A=\sum_j \alpha_j P_j$ where $\alpha_j \geqq 0$, $\sum_j \alpha_j=1$ and P_j is a sub-permutation matrix of rank $n-i$. Then each matrix $\alpha_j P_j$ has the sum of all its entries equal to $(n-i)\alpha_j$ and each row or column sum has the value α_j or 0. Hence $S=(n-i)\sum_j \alpha_j=(n-i)$ and $M \leqq \sum_j \alpha_j=1$. Also since $n-i=S=\sum_j R_j \leqq nM$, $(n-i)/n \leqq M$. Hence $S=n-i$ and $(n-i)/n \leqq M \leqq 1$.

To obtain the sufficiency we note that if $S=n-i$ and $(n-i)/n$

Received by the editors July 25, 1957.

[1] The research reported upon here was supported in part by the National Research Council of Canada and the Mathematical Institute of the Canadian Congress of Mathematics.

$\leqq M \leqq 1$ then $\sum R_j = \sum C_j = n - i$. Also the numbers $1 - R_1$, $1-R_2, \cdots, 1-R_n$ are non-negative and at least one of these is positive if $i>0$. For if all of $1-R_1$, $1-R_2, \cdots, 1-R_n$ were 0 then $R_j=1=M$ for all j so that $S=n$ a contradiction. The matrix A is now augmented to a matrix $A^\star$ by the addition of i rows and i columns as follows: $a_{rs}^\star = a_{rs}$ if r and s are both less than or equal to n; $a_{rs}^\star=0$ if r and s are both greater than n; $a_{r,n+t}^\star=(1-R_r)/i$ for $r=1, 2, \cdots, n; t=1, 2, \cdots, i$; $a_{n+u,v}^\star=(1-C_v)/i$ for $u=1, 2, \cdots, i$; $v=1, 2, \cdots, n$. The matrix $A^\star$ is a doubly stochastic $n+i$ by $n+i$ matrix with zeros in the lower right hand i by i block. By the theorem quoted in the introduction $A^\star = \sum \alpha_r P_r^\star$ where $\alpha_r \geqq 0$, $\sum \alpha_r = 1$ and $P_r^\star$ is an $n+i$ by $n+i$ permutation matrix. Furthermore, each $P_r^\star$ has an i by i block of zeros in its lower right corner. Hence $P_r^\star$ has $2i$ entries equal to 1 in its last i rows and i columns. If P_r is the n by n matrix in the upper left hand corner of $P_r^\star$, P_r contains $(n+i)-2i$ $=n-i$ ones. Hence P_r, is a sub-permutation matrix of rank $n-i$. Also $A = \sum \alpha_r P_r$.

BIBLIOGRAPHY

1. L. Dulmage and I. Halperin, *On a theorem of Fröbenius-Konig and J. von Neumann's game of hide and seek*, Transactions of the Royal Society of Canada. Section III. vol. 49 (1955) pp. 23–29.

2. L. Dulmage and N. S. Mendelsohn, *Some generalizations of the problem of distinct representatives*, Canadian J. Math. (to appear).

→ H. B. Mann and H. J. Ryser, *Systems of distinct representatives*, Amer. Math. Monthly vol. 60 (1953) pp. 397–401.

UNIVERSITY OF MANITOBA AND
MATHEMATICS INSTITUTE, QUEEN'S UNIVERSITY